Discrete Probability - Fundamentals

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Outline

1. Motivation

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2. Preliminaries
   - Sample Space and Events
   - Defining Probabilities on Events

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Note

Discrete Probability combines aspects of combinatorics, logic and inference.

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Any subset of the sample space S is called an event.

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(iii) A probability function, satisfying the properties discussed previously.

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In the die tossing experiment, what is the probability of the event $\{2, 4, 6\}$?

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Some more examples

Example

A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy? Assume that the sample space is $S = \{(b, g),\ (b, b),\ (g, b),\ (g, g)\}$ and that all outcomes are equally likely.

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Assume that an urn contains 7 *black balls and* 5 *white balls. Two balls are chosen from this urn, one after the other, without replacement and at random. What is the probability that both balls are black?*

Solution

Let E denote the event that the first ball is black and F denote the event that the second ball is black. Clearly, we are interested in $P(EF)$. Observe that $P(E) = \frac{7}{12}$ and $P(F \mid E) = \frac{6}{11}$. Now, $P(F \mid E) = \frac{P(EF)}{P(E)}$, and hence,

Some more examples

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Definition

Two events E and F on a sample space S are said to be independent, if the occurrence of one does not affect the occurrence of the other.

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Bayes' Formula

Derivation

Let E and F denote two arbitrary events on a sample space S.

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Let E and F denote two arbitrary events on a sample space S. Clearly, $E = EF \cup EF^c$, where the events EF and EF^c are mutually exclusive. Now, observe that,

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Thus, the probability of an event E is the weighted average of the conditional probability of E, given that event F has occurred and the conditional probability of E, given that event F has not occurred, each conditional probability being given as much weight as the probability of the event that it is conditioned on, has of occurring.

One Final Example

Example

Consider two urns.

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Consider two urns. Urn 1 has 2 white balls and 7 black balls.

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Therefore, $P(H \mid W) = \frac{\frac{1}{9}}{\frac{67}{198}}$

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Therefore, $P(H \mid W) = \frac{\frac{1}{9}}{\frac{67}{198}} = \frac{22}{67}$,

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Therefore, $P(H \mid W) = \frac{\frac{1}{9}}{\frac{67}{198}} = \frac{22}{67}$, i.e., the conditional probability that the ball was drawn from Urn 1, given that it is white, is $\frac{22}{67}$.

Identities

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Intersection Lemma

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Let E_1, E_2, ..., E_n denote n events. Then,

$$P(\cap_{i=1}^{n} E_i) = P(E_1) \cdot P(E_2 \mid E_1) \cdot P(E_3 \mid E_1 \cap E_2) \ldots \cdot P(E_n \mid \cap_{i=1}^{n-1} E_i)$$

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Let E_1, E_2, ..., E_n denote n events. Then,

$$P(\cup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} P(E_i)$$