Randomized Algorithms - Homework I

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1 Instructions

1. The homework is due on February 2, in class.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.
4. The work must be entirely your own. You are expressly **prohibited** from consulting with colleagues or the internet (with the exception of the material on the course website). You may refer to books in the library, crediting the same, if used in your work.

2 Problems

1. Let E_1, E_2, . . ., E_n denote n events defined on some probability space. Show that
$$P(\cap_{i=1}^{n} E_i) = P(E_1) \cdot P(E_2 \mid E_1) \cdot P(E_3 \mid E_1 \cap E_2) \ldots \cdot P(E_n \mid \cap_{i=1}^{n-1} E_i).$$
2. Consider the following balls and bins game. You are given a bin containing a white ball and a black ball. The game proceeds in a series of rounds. In each round, we draw a ball, uniformly and at random from the bin. If the ball drawn is a white ball, we add two white balls to the bin; otherwise, we add two black balls to the bin. The game concludes when the bin contains exactly n balls. (Note that in each round, the number of balls in the bin increases by exactly one, since one ball is taken out from the bin and two are added to it.) Argue that the number of white balls in the bin is equally likely to be any number between 1 and $(n-1)$.
3. Suppose that an airplane engine will fail in flight with probability $(1-p)$, independently from engine to engine. A flight will be successful if at least 50% of its engines remain operative in flight. For what values of p is a four-engine airplane preferable to a two-engine airplane?
4. A miner is trapped in a mine containing three doors. The first door leads to a tunnel that will allow the miner to reach safety in two hours. The second door leads to a tunnel that will bring him back to the mine after three hours. The third door leads to a tunnel that will bring him back to the mine after five hours. Assume that the miner chooses one of the three doors uniformly and at random, whenever he is confronted with the three choices. In how much time can he expect to get to safety?
5. Compute the variance of the geometric random variable with parameter p, using the method of conditional expectations (see page 32 of [MU05]).

References

[MU05] Michael Mitzenmacher and Eli Upfal. *Probability and Computing*. Cambridge University Press, 1^{st} edition, 2005.