Edmonds-Giles theorem

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Outline

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Fundamentals

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Definition

A polyhedron in $\mathbb{R}^n$ is a set of the form: $P = \{\mathbf{x} \in \Re^n \ : \ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\}$, for some matrix $\mathbf{A} \in \Re^{m \times n}$ and vector $\mathbf{b} \in \Re^m$.

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Lemma

Let $\mathbf{A}$ be a rational matrix and let $\mathbf{b}$ be a rational column vector. Then, $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has an integral solution if and only if $\mathbf{y} \cdot \mathbf{b}$ is an integer, for each each rational vector $\mathbf{y}$, for which $\mathbf{y} \cdot \mathbf{A}$ is integral.

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A polyhedral system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ is said to be **Totally Dual Integral** (TDI), if the minimum in the Linear Programming duality equation

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TDI-theorem Let $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ *denote a TDI system, where* $\mathbf{A}$ *is rational and* $\mathbf{b}$ *is integral.*

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TDI-theorem Let $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ *denote a TDI system, where* $\mathbf{A}$ *is rational and* $\mathbf{b}$ *is integral. Then the polyhedron* $\{\mathbf{x} \; : \; \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\}$ *is integral.*

Totally Dual Integral Systems

Definition

A polyhedral system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ is said to be **Totally Dual Integral** (TDI), if the minimum in the Linear Programming duality equation

$$\max\{\mathbf{c} \cdot \mathbf{x} \,:\, \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\} = \min\{\mathbf{y} \cdot \mathbf{b} \,:\, \mathbf{y} \cdot \mathbf{A} = \mathbf{c},\ \mathbf{y} \geq \mathbf{0}\}$$

has an integral solution $\mathbf{y}$ for each integral $\mathbf{c}$, for which the minimum is finite.

Theorem

TDI-theorem Let $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ *denote a TDI system, where* $\mathbf{A}$ *is rational and* $\mathbf{b}$ *is integral. Then the polyhedron* $\{\mathbf{x} \,:\, \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\}$ *is integral.*

Proof.

Direct consequence of $(g) \Rightarrow (a)$! □

Hoffman-Kruskal Theorem

Hoffman-Kruskal Theorem

Theorem

An integral matrix $\mathbf{A}$ *is totally unimodular (TU), if and only if the polyhedron* $\{\mathbf{x} \,:\, \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\}$ *is integral for each integral vector* $\mathbf{b}$*.*

Homework II, Problem 5

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Theorem

An integral matrix $\mathbf{A}$ *is totally unimodular if and only if the system* $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *is TDI for each vector* $\mathbf{b}$*.*

Homework II, Problem 5

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An integral matrix $\mathbf{A}$ *is totally unimodular if and only if the system* $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *is TDI for each vector* $\mathbf{b}$*.*

Proof.

Only If: Assume that $\mathbf{A}$ is TU.

Homework II, Problem 5

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An integral matrix $\mathbf{A}$ *is totally unimodular if and only if the system* $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *is TDI for each vector* $\mathbf{b}$*.*

Proof.

Only If: Assume that $\mathbf{A}$ is TU. It follows that $\mathbf{A}^{\mathbf{T}}$ is TU.

Homework II, Problem 5

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An integral matrix $\mathbf{A}$ *is totally unimodular if and only if the system* $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b},\, \mathbf{x} \geq \mathbf{0}\}$ *is TDI for each vector* $\mathbf{b}$*.*

Proof.

Only If: Assume that $\mathbf{A}$ is TU. It follows that $\mathbf{A}^{\mathbf{T}}$ is TU. Hence, $\min\{\mathbf{y} \cdot \mathbf{b} \,:\, \mathbf{y} \cdot \mathbf{A} \geq \mathbf{c},\, \mathbf{y} \geq \mathbf{0}\}$ is attained by an integral vector, for each vector $\mathbf{b}$ and each integral vector $\mathbf{c}$, for which the minimum is finite

Homework II, Problem 5

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An integral matrix $\mathbf{A}$ *is totally unimodular if and only if the system* $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ *is TDI for each vector* $\mathbf{b}$*.*

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Homework II, Problem 5

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An integral matrix **A** *is totally unimodular if and only if the system* $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *is TDI for each vector* **b***.*

Proof.

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Homework II, Problem 5

Theorem

An integral matrix $\mathbf{A}$ *is totally unimodular if and only if the system* $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$ *is TDI for each vector* $\mathbf{b}$*.*

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If: Assume that the system $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$ is TDI for each vector $\mathbf{b}$.

Homework II, Problem 5

Theorem

An integral matrix $\mathbf{A}$ *is totally unimodular if and only if the system* $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ *is TDI for each vector* $\mathbf{b}$*.*

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If: Assume that the system $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ is TDI for each vector $\mathbf{b}$. It follows that the system is TDI for each integral vector $\mathbf{b}$.

Homework II, Problem 5

Theorem

An integral matrix $\mathbf{A}$ *is totally unimodular if and only if the system* $\{\mathbf{A} \cdot \mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}\}$ *is TDI for each vector* $\mathbf{b}$*.*

Proof.

Only If: Assume that $\mathbf{A}$ is TU. It follows that $\mathbf{A^T}$ is TU. Hence, $\min\{\mathbf{y} \cdot \mathbf{b}\ :\ \mathbf{y} \cdot \mathbf{A} \ge \mathbf{c},\ \mathbf{y} \ge \mathbf{0}\}$ is attained by an integral vector, for each vector $\mathbf{b}$ and each integral vector $\mathbf{c}$, for which the minimum is finite (Hoffman-Kruskal)! In other words, the system $\{\mathbf{A} \cdot \mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}\}$ is TDI, for each vector $\mathbf{b}$.

If: Assume that the system $\{\mathbf{A} \cdot \mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}\}$ is TDI for each vector $\mathbf{b}$. It follows that the system is TDI for each integral vector $\mathbf{b}$. By the TDI-theorem, the polyhedron $\{\mathbf{x}\ :\ \mathbf{A} \cdot \mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}\}$ is integral for each integral vector $\mathbf{b}$.

Homework II, Problem 5

Theorem

An integral matrix $\mathbf{A}$ *is totally unimodular if and only if the system* $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$ *is TDI for each vector* $\mathbf{b}$*.*

Proof.

Only If: Assume that $\mathbf{A}$ is TU. It follows that $\mathbf{A}^\mathbf{T}$ is TU. Hence, $\min\{\mathbf{y} \cdot \mathbf{b} \ : \ \mathbf{y} \cdot \mathbf{A} \geq \mathbf{c}, \ \mathbf{y} \geq \mathbf{0}\}$ is attained by an integral vector, for each vector $\mathbf{b}$ and each integral vector $\mathbf{c}$, for which the minimum is finite (Hoffman-Kruskal)! In other words, the system $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$ is TDI, for each vector $\mathbf{b}$.

If: Assume that the system $\{\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$ is TDI for each vector $\mathbf{b}$. It follows that the system is TDI for each integral vector $\mathbf{b}$. By the TDI-theorem, the polyhedron $\{\mathbf{x} \ : \ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}$ is integral for each integral vector $\mathbf{b}$. By the Hoffman-Kruskal theorem, it follows that $\mathbf{A}$ is TU.

□