Foundations of the Simplex Method

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Outline

1. Graphical solutions to two dimensional problems
   - Representing constraints as sections of the plane
   - Handling the objective function
   - Exercises

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Geometrically representing linear programs

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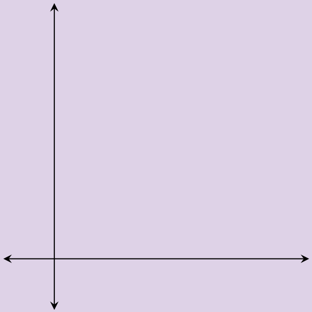

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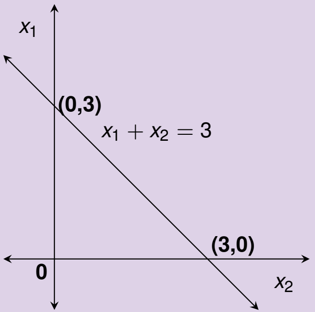

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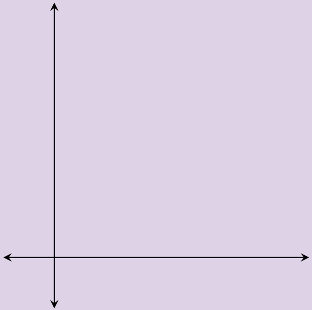

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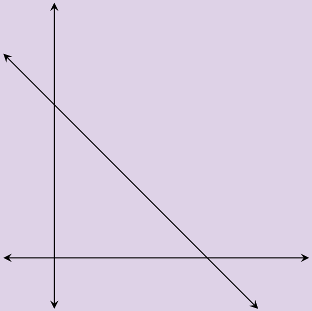

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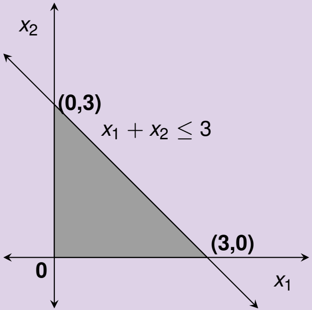
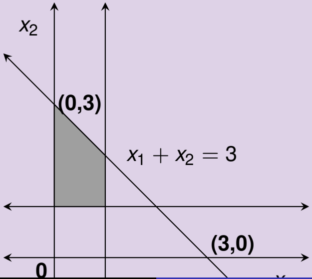


Geometric View of Constraints

For a system of constraints the section of the plane corresponding to solutions to that system is simply the intersection of the portions of the plane corresponding to each constraint. For instance, the constraints

$$\begin{aligned} x_1 &\leq 1 \\ x_2 &\geq 1 \\ x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

would produce



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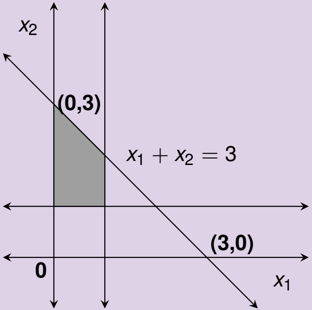
If we are trying to maximize z then we find the maximum z for which the corresponding line passes though the portion of the plane corresponding to the system of constraints. When minimizing we consider the minimum such z.
It also helps to find the gradient of z as it identifies the direction in which z grows the fastest. As the objective function is of the form $z = c_1 x_1 + c_2 x_2$, the gradient is simply the vector (c_1, c_2).

Handling the objective function

For example adding the objective function $z = x_1 + 2x_2$ to our previous example yields



Thus trying to maximize z would yield that maximum z to be 6 when $x_1 = 0$ and $x_2 = 3$. Similarly trying to minimize z would yield that minimum z to be 2 when $x_1 = 0$ and $x_2 = 1$.

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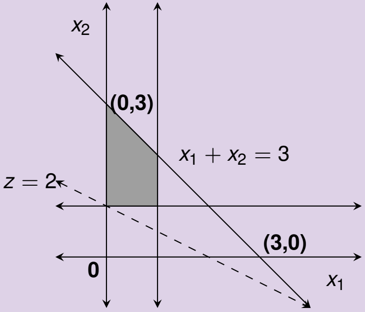

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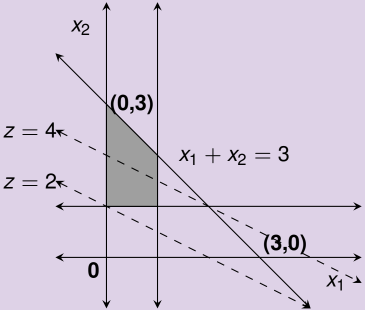

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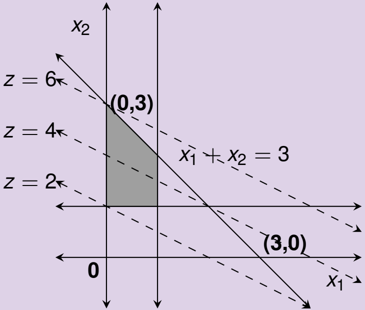

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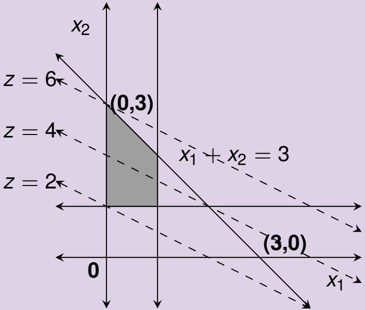

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Exercise 1

Solve the following linear program graphically

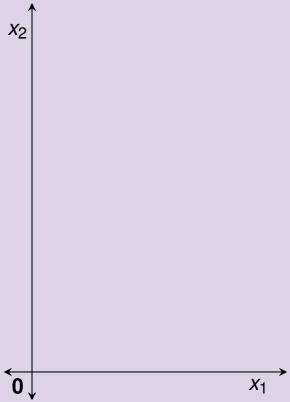
$$\text{minimize } z = 4x_1 + 5x_2$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 &\leq 24 \\ x_1 &\geq 5 \\ 3x_1 - x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution

If the constraints are plotted onto a graph we see



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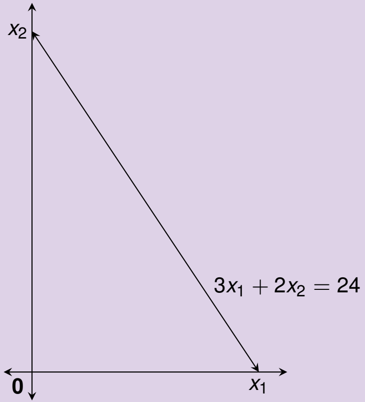

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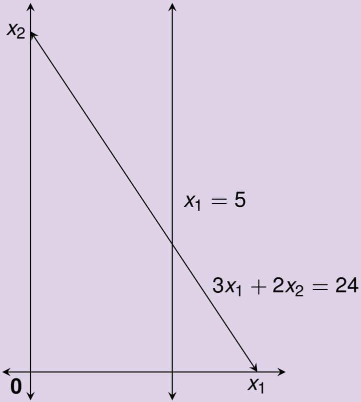

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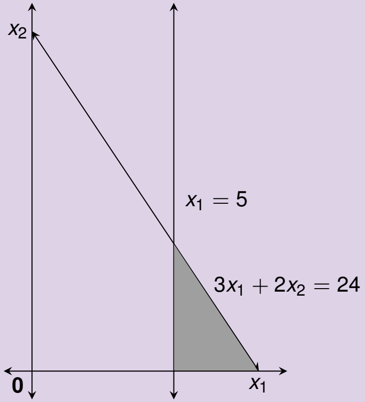

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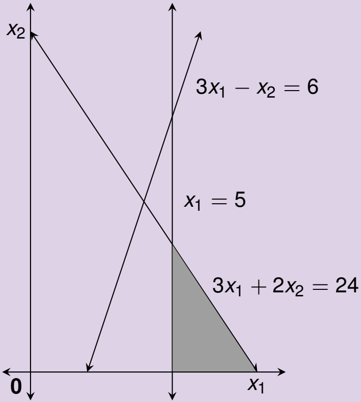

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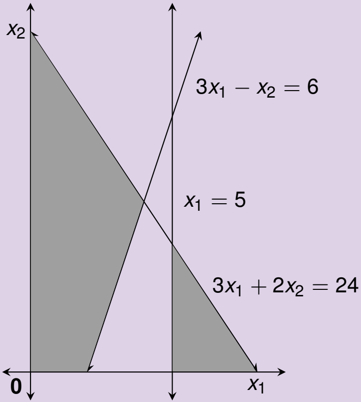

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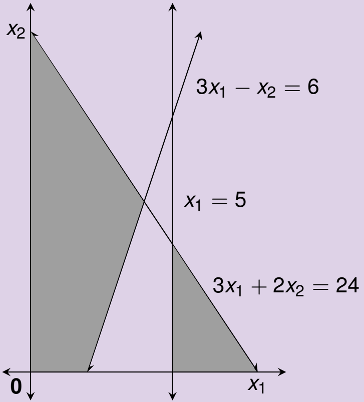

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Exercise 2

Solve the following system of constraints graphically

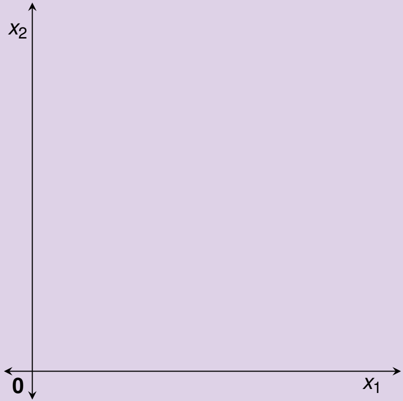
$$\text{minimize } z = x_1 - 4x_2$$

subject to

$$\begin{array}{rcl} x_1 + x_2 & \leq & 12 \\ -2x_1 + x_2 & \leq & 4 \\ x_2 & \leq & 8 \\ x_1 - 3x_2 & \leq & 4 \\ x_1, x_2 & \geq & 0 \end{array}$$

Solution

Plotting the constraints and then checking various values of z we get.



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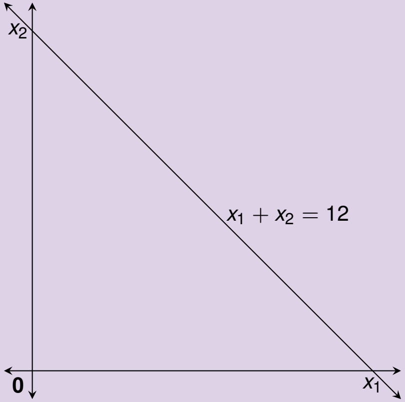

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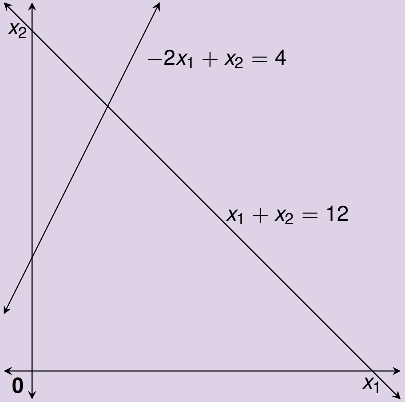

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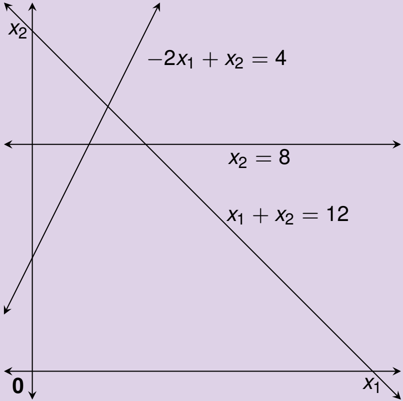

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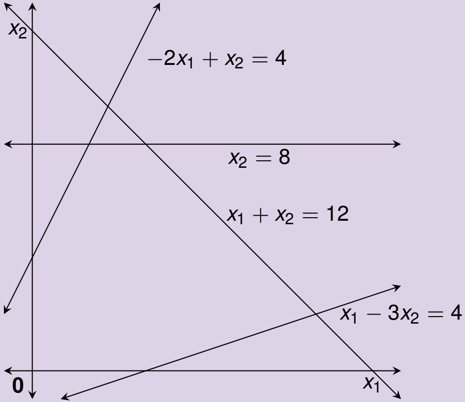

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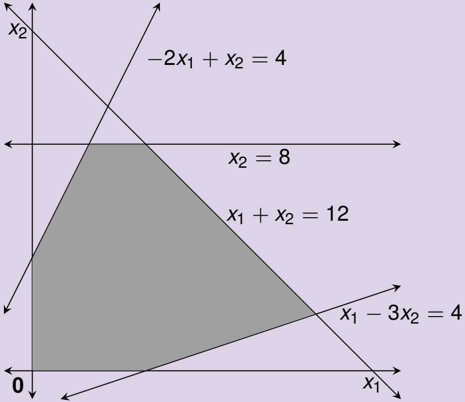

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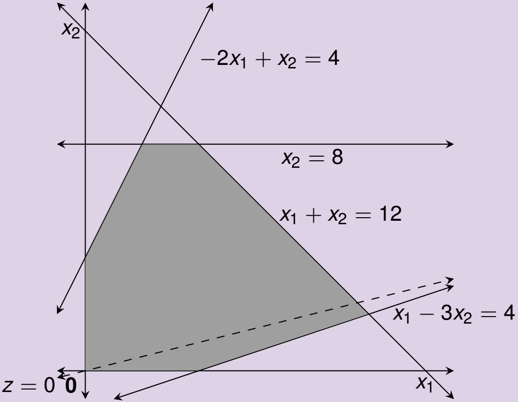

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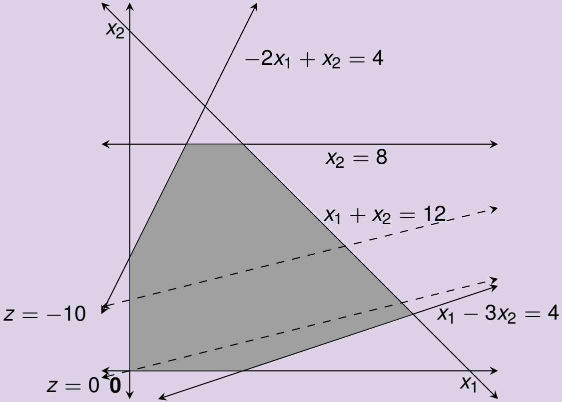

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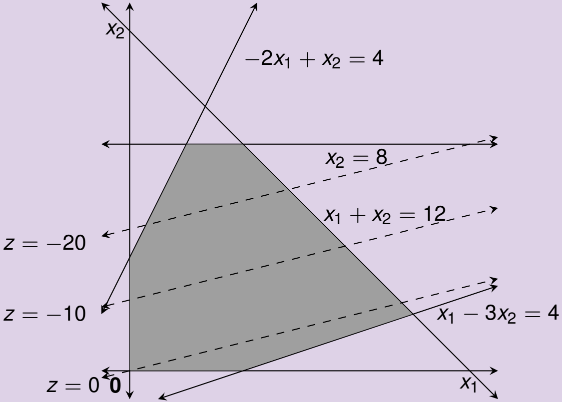

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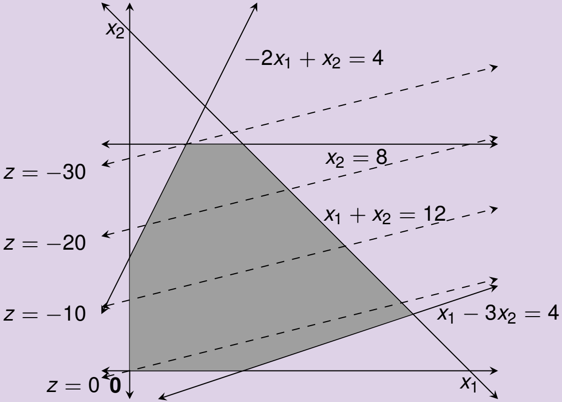

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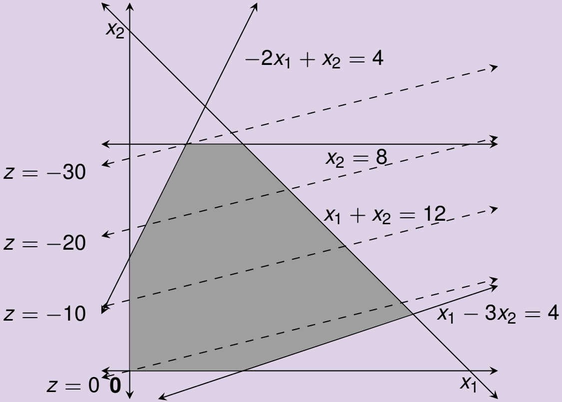

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Exercise 3

Solve the following linear program graphically

$$\text{maximize } z = x_1 + 2x_2$$

subject to

$$\begin{aligned}
-2x_1 + x_2 &\leq 2 \\
2x_1 + 5x_2 &\geq 10 \\
x_1 - 4x_2 &\leq 2 \\
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Solution

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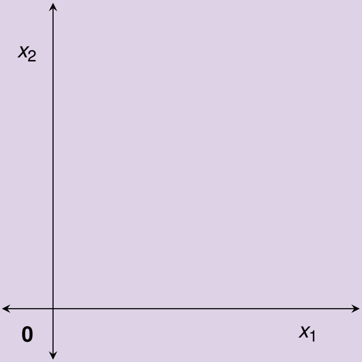

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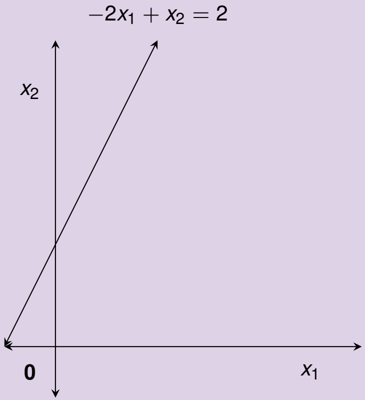

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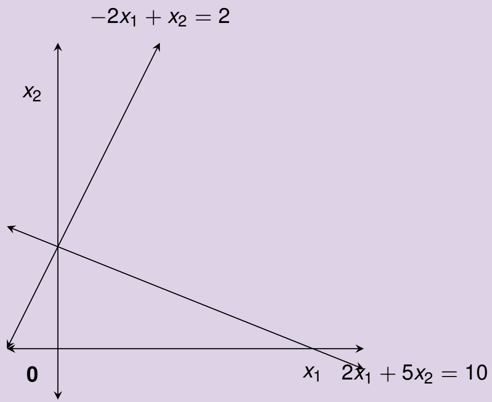

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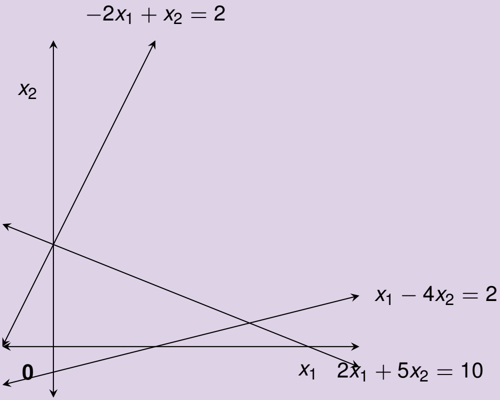

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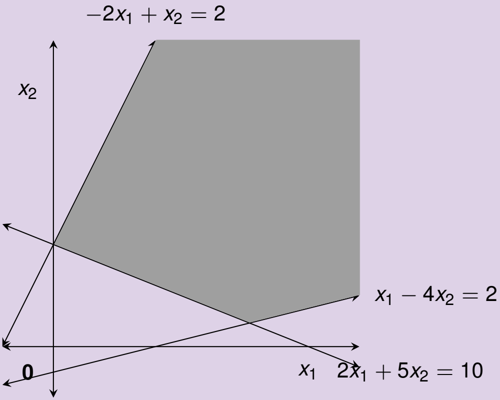

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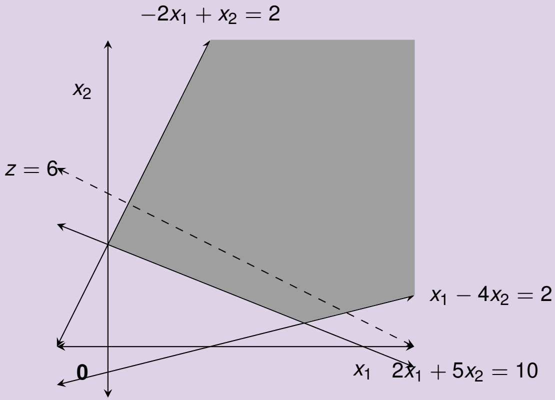

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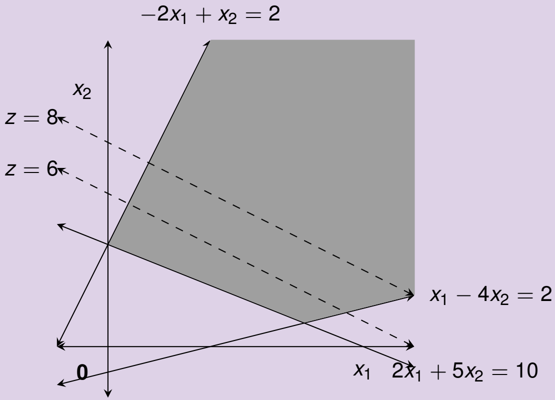

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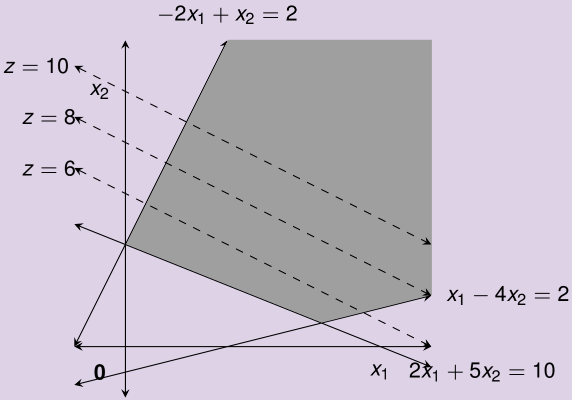

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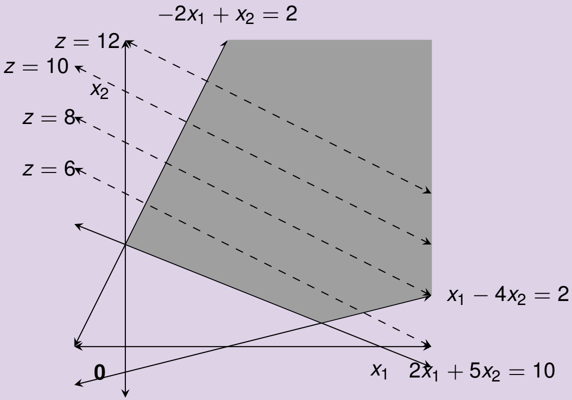

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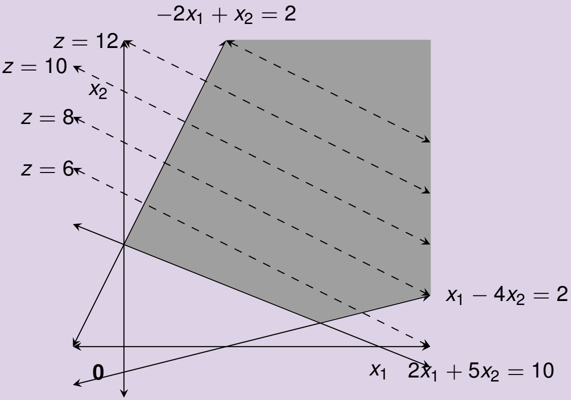

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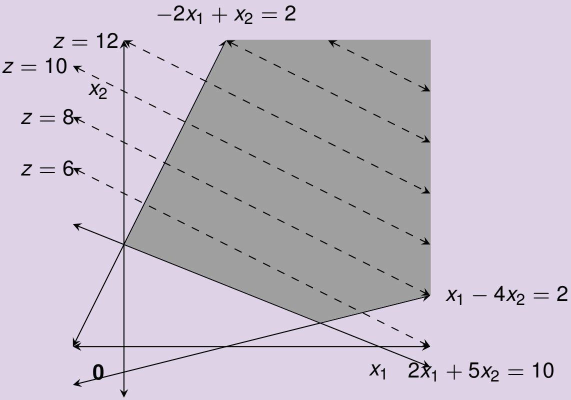

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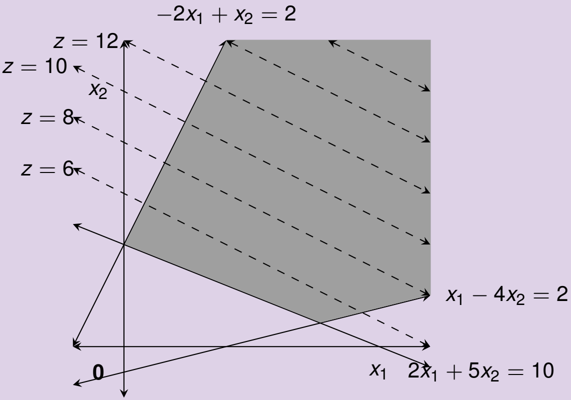

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Definition (Hyperplane)

A hyperplane is a set of points, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$, that satisfy $\mathbf{ax} = b$, where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and b is a scalar.

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Definition (Halfspace)

A closed halfspace corresponding to a hyperplane $\mathbf{a} \cdot \mathbf{x} = b$ is either of the sets $H^+ = \{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} \geq b\}$ or $H^- = \{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} \leq b\}$. If the inequalities involved are strict then the corresponding halfspace are referred to as open halfspaces.

Outline

Definition (Convex Set)

A set, S, is convex if for any two points $\mathbf{x}_1, \mathbf{x}_2 \in S$ then all points on the line segment connecting $\mathbf{x}_1$ and $\mathbf{x}_2$ are in S. This means that $\forall \alpha \in [0, 1]$, $\alpha \cdot \mathbf{x}_1 + (1 - \alpha) \cdot \mathbf{x}_2 \in S$.

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Systems of constraints as Polyhedral Sets

A constraint system of the form $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is a polyhedral set as each constraint corresponds to a halfspace.

Theorem

The set $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *is convex.*

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Proof.

Let $\mathbf{x}_1, \mathbf{x}_2 \in S$, and let $\alpha \in [0, 1]$.

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Let $\mathbf{x}_1, \mathbf{x}_2 \in S$, and let $\alpha \in [0, 1]$. So, by definition of S, $\mathbf{A} \cdot \mathbf{x}_1 = \mathbf{b}$ and $\mathbf{A} \cdot \mathbf{x}_2 = \mathbf{b}$.
Thus for any $\alpha \in [0, 1]$, we have that
$\mathbf{A} \cdot (\alpha \cdot \mathbf{x}_1 + (1 - \alpha) \cdot \mathbf{x}_2) = \alpha \cdot \mathbf{A} \cdot \mathbf{x}_1 + (1 - \alpha) \cdot \mathbf{A} \cdot \mathbf{x}_2 = \alpha \cdot \mathbf{b} + (1 - \alpha) \cdot \mathbf{b} = \mathbf{b}$.

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Let $\mathbf{x}_1, \mathbf{x}_2 \in S$, and let $\alpha \in [0, 1]$. So, by definition of S, $\mathbf{A} \cdot \mathbf{x}_1 = \mathbf{b}$ and $\mathbf{A} \cdot \mathbf{x}_2 = \mathbf{b}$. Thus for any $\alpha \in [0, 1]$, we have that $\mathbf{A} \cdot (\alpha \cdot \mathbf{x}_1 + (1 - \alpha) \cdot \mathbf{x}_2) = \alpha \cdot \mathbf{A} \cdot \mathbf{x}_1 + (1 - \alpha) \cdot \mathbf{A} \cdot \mathbf{x}_2 = \alpha \cdot \mathbf{b} + (1 - \alpha) \cdot \mathbf{b} = \mathbf{b}$. And as $\alpha, (1 - \alpha) \geq 0$ and $\mathbf{x}_1, \mathbf{x}_2 \geq \mathbf{0}$ we have that $\alpha \cdot \mathbf{x}_1 + (1 - \alpha) \cdot \mathbf{x}_2 \geq \mathbf{0}$. Thus for any $\alpha \in [0, 1]$ $\alpha \cdot \mathbf{x}_1 + (1 - \alpha) \cdot \mathbf{x}_2 \in S$. Thus S is convex. $\square$

Outline

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A point **x** in a convex set S is said to be an extreme point if it does not lie on the interior of a line segment connecting two distinct points in S.

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Definition (Direction)

A non-zero vector $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is a direction of a convex set S if $\forall \mathbf{x} \in S$ and $\forall \lambda \geq 0$, $\mathbf{x} + \lambda \cdot \mathbf{d} \in S$.

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Definition (Extreme Direction)

A direction $\mathbf{d}$ of a convex set S is said to be an extreme direction if cannot be expressed as a positive combination of two distinct directions of S.

Outline

Goal

First we want to develop a method of identifying the extreme points of a system of constraints in standard form.

Theorem

Let $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, where $\mathbf{A}$ is $m \times n$ and rank$(\mathbf{A}) = m < n$. $\bar{\mathbf{x}}$ is an extreme point of S if and only if $\bar{\mathbf{x}}$ is the intersection of n linearly independent hyperplanes.

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only if.

As $\mathbf{Q}$ is an $(m+p) \times n$ matrix where $m+p < n$ the columns of $\mathbf{Q}$ are linearly dependent. Thus there exists $\mathbf{y} \neq \mathbf{0}$ such that $\mathbf{Q}\mathbf{y} = \mathbf{0}$.

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Recognizing directions

Now that we have a different way of recognizing extreme points we will also determine a way of recognizing directions.

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Theorem

Let $S = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. *Then* $\mathbf{d}$ *is a direction of* S *if and only if* $\mathbf{d} \in D = \{\mathbf{d} : \mathbf{A}\mathbf{d} = \mathbf{0}, \mathbf{d} \geq \mathbf{0}, \mathbf{d} \neq \mathbf{0}\}$.

only if.

Let $\mathbf{d}$ be a direction of S. Thus $\mathbf{d} \neq \mathbf{0}$ and $\mathbf{x} + \lambda\mathbf{d} \in S$ for all $\mathbf{x} \in S$ and $\lambda \geq 0$.

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Let $\mathbf{d}$ be a direction of S. Thus $\mathbf{d} \neq \mathbf{0}$ and $\mathbf{x} + \lambda\mathbf{d} \in S$ for all $\mathbf{x} \in S$ and $\lambda \geq 0$. Thus, for $\lambda = 1$, we have that $\mathbf{b} = \mathbf{A}(\mathbf{x} + \mathbf{d}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{d} = \mathbf{b} + \mathbf{A}\mathbf{d}$. Thus $\mathbf{A}\mathbf{d} = \mathbf{0}$.

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Let $\mathbf{d}$ be a direction of S. Thus $\mathbf{d} \neq \mathbf{0}$ and $\mathbf{x} + \lambda\mathbf{d} \in S$ for all $\mathbf{x} \in S$ and $\lambda \geq 0$. Thus, for $\lambda = 1$, we have that $\mathbf{b} = \mathbf{A}(\mathbf{x} + \mathbf{d}) = \mathbf{Ax} + \mathbf{Ad} = \mathbf{b} + \mathbf{Ad}$. Thus $\mathbf{Ad} = \mathbf{0}$. We also have that $\mathbf{x} + \lambda\mathbf{d} \geq 0$. This implies that $\mathbf{d} \geq \mathbf{0}$ because otherwise λ could be made large enough to make $\mathbf{x} + \lambda\mathbf{d} < \mathbf{0}$.

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Let $\mathbf{d} \in D$ and let $\mathbf{x} \in S$. We have that, by definition of D, $\mathbf{d} \neq \mathbf{0}$.

only if.

Let $\mathbf{d}$ be a direction of S. Thus $\mathbf{d} \neq \mathbf{0}$ and $\mathbf{x} + \lambda\mathbf{d} \in S$ for all $\mathbf{x} \in S$ and $\lambda \geq 0$. Thus, for $\lambda = 1$, we have that $\mathbf{b} = \mathbf{A}(\mathbf{x} + \mathbf{d}) = \mathbf{Ax} + \mathbf{Ad} = \mathbf{b} + \mathbf{Ad}$. Thus $\mathbf{Ad} = \mathbf{0}$. We also have that $\mathbf{x} + \lambda\mathbf{d} \geq 0$. This implies that $\mathbf{d} \geq \mathbf{0}$ because otherwise λ could be made large enough to make $\mathbf{x} + \lambda\mathbf{d} < \mathbf{0}$. Thus $\mathbf{d} \in D$. $\square$

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Let $\mathbf{d} \in D$ and let $\mathbf{x} \in S$. We have that, by definition of D, $\mathbf{d} \neq \mathbf{0}$. Also by the definition of D we have that $\mathbf{Ad} = \mathbf{0}$, thus for $\lambda \geq 0$ we have that $\mathbf{A}(\mathbf{x} + \lambda\mathbf{d}) = \mathbf{Ax} + \mathbf{Ad} = \mathbf{b}$ also as $\mathbf{x}, \mathbf{d} \geq \mathbf{0}$ we have that $\mathbf{x} + \lambda\mathbf{d} \geq \mathbf{0}$.

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Bounded and unbounded sets

Before continuing on, we define what it it means for a set, specifically a subset of $\mathbb{R}^n$ to be either bounded or unbounded.

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Definition (Bounded Set)

A subset S of $\mathbb{R}^n$ is bounded if it can be contained within an n-dimensional ball. Mathematically S is bounded if there exist a point $\bar{\mathbf{x}} \in S$ and a radius $r > 0$ such that for any point $\mathbf{x} \in S$ the distance from $\bar{\mathbf{x}}$ to $\mathbf{x}$ is less than r.

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Definition (Unbounded Set)

An unbounded set is simply a set which is not bounded.

Outline

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1. *S has at least one extreme point and at most a finite number of extreme points, thus* $E = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \neq \emptyset$.

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1. *S has at least one extreme point and at most a finite number of extreme points, thus $E = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \neq \emptyset$.*
2. *S is unbounded of and only if S has at least one extreme direction.*

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3. *if S is unbounded then S has a finite number of extreme directions, $D = \{\mathbf{d}_1, \ldots, \mathbf{d}_q\} \neq \emptyset$.*

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3. *if S is unbounded then S has a finite number of extreme directions, $D = \{\mathbf{d}_1, \ldots, \mathbf{d}_q\} \neq \emptyset$.*
4. *if $\mathbf{x} \in S$, then $\mathbf{x}$ can be written as a convex combination of extreme points plus a positive combination of extreme vectors, that is $\mathbf{x} = \sum_{i=1}^{p}(\alpha_i \mathbf{x}_i) + \sum_{j=1}^{q}(\lambda_j \mathbf{d}_j)$ for $\sum_{i=1}^{p} \alpha_i = 1$, $\alpha_i \geq 0$, $\lambda_j \geq 0$.*

Theorem

Let $S = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *and consider the following linear program.*

$$\begin{array}{c} \text{maximize } z = \mathbf{c}\mathbf{x} \\ \text{subject to } \mathbf{x} \in S. \end{array}$$

Suppose S *is unbounded and has extreme points* $E = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \neq \emptyset$ *and extreme directions* $D = \{\mathbf{d}_1, \ldots, \mathbf{d}_q\} \neq \emptyset$*. Let* z^* *represent the optimal objective value of the linear program. Then* z^* *is finite if and only if* $\mathbf{c}\mathbf{d}_j \leq 0$ *for all* $\mathbf{d}_j \in D$*. And if a finite optimal solution exists then an extreme point optimal solution exists.*

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Proof.

By the fourth part of the previous theorem we can express any $\mathbf{x} \in S$ in terms of extreme points and extreme directions. Thus we can modify the linear program to

$$\begin{array}{c} \text{maximize } z = \sum_{i=1}^{p}(\alpha_i \mathbf{c}\mathbf{x}_i) + \sum_{j=1}^{q}(\lambda_j \mathbf{c}\mathbf{d}_j) \\ \text{subject to } \sum_{i=1}^{p} \alpha_i = 1,\ \alpha_i \geq 0,\ \lambda_j \geq 0. \end{array}$$

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Suppose, to obtain a contradiction, that for some $\mathbf{d}_j \in D$ $\mathbf{cd} > 0$. However $\lambda_j \mathbf{cd}$ is a term of z thus z can be made arbitrarily large by making λ_j arbitrarily large.

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Suppose, to obtain a contradiction, that for some $\mathbf{d}_j \in D$ $\mathbf{cd} > 0$. However $\lambda_j \mathbf{cd}$ is a term of z thus z can be made arbitrarily large by making λ_j arbitrarily large. This contradicts that z^* is finite and so $\mathbf{cd} \leq 0$ for all $\mathbf{d}_j \in D$. $\square$

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As there are finitely many extreme points there must be an extreme point $\mathbf{x}_k$ for which $(cx)_k \geq (cx)_i$ for all $i = 1, \ldots, p$.

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As there are finitely many extreme points there must be an extreme point $\mathbf{x}_k$ for which $(cx)_k \geq (cx)_i$ for all $i = 1, \ldots, p$. Thus we have that $z = \sum_{i=1}^{p} (\alpha_i \mathbf{cx}_i) \leq \sum_{i=1}^{p} (\alpha_i \mathbf{cx}_k) = \mathbf{cx}_k$. Thus z^* is finite and in fact $z^* = \mathbf{cx}_k$ where $\mathbf{x}_k$ is the optimal extreme point. □

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Finding basic feasible solutions

Consider a linear system of equations $\mathbf{Ax} = \mathbf{b}$. Where $\mathbf{A}$ is an $m \times n$ matrix $\mathbf{b} = (b_1, \ldots, b_m)^t$, and $\mathbf{x} = (x_1, \ldots, x_n)^t$.

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Finding basic feasible solutions

Thus we can rewrite $\mathbf{Ax} = \mathbf{b}$ as $\mathbf{Bx}_B + \mathbf{Nx}_N = \mathbf{b}$ where $\mathbf{x} = \binom{\mathbf{x}_B}{\mathbf{x}_N}$.

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Thus we can rewrite $\mathbf{Ax} = \mathbf{b}$ as $\mathbf{Bx}_B + \mathbf{Nx}_N = \mathbf{b}$ where $\mathbf{x} = \binom{\mathbf{x}_B}{\mathbf{x}_N}$. As $\mathbf{B}$ is non-singular the inverse of $\mathbf{B}$ exists. Thus we have that $\mathbf{B}^{-1}\mathbf{Bx}_B + \mathbf{B}^{-1}\mathbf{Nx}_N = \mathbf{B}^{-1}\mathbf{b}$. Which is equivalent to sating that $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{Nx}_N$.

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Outline

Theorem

Let $S = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$*, where* $\mathbf{A}$ *is* $m \times n$ *and* $\text{rank}(\mathbf{A}) = m < n$*.* $\mathbf{x}$ *is an extreme point of* S *if and only if* $\mathbf{x}$ *is a basic feasible solution.*

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Let $\mathbf{x} \in S$ be an extreme point. Thus $\mathbf{x}$ is the intersection of n linearly independent hyperplanes. From the definition of S the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ provide m of these hyperplanes. Thus the remaining $n - m$ hyperplanes must come from the non-negativity constraints.

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Thus $\mathbf{x}_B$ is the unique solution to $\mathbf{B}\mathbf{x}_B = \mathbf{b}$. Thus $\mathbf{B}$ is invertible and a basis matrix. Thus $\mathbf{x} \geq 0$ is a basic feasible solution. □

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All together

All together this gives us the basics of a method for finding the optimal solutions to a linear program.