The Simplex Algorithm

K. Subramani[1]

[1] Lane Department of Computer Science and Electrical Engineering
West Virginia University

4 February, 2013

Outline

Outline

Overview

Topics

Overview

Topics

(i) Unique optimal solution, alternative optimal solutions, and unbounded objective.

Overview

Topics

(i) Unique optimal solution, alternative optimal solutions, and unbounded objective.
(ii) If a finite optimal solution exists, then an extreme-point optimal solution exists.

Overview

Topics

(i) Unique optimal solution, alternative optimal solutions, and unbounded objective.

(ii) If a finite optimal solution exists, then an extreme-point optimal solution exists.

(iii) Each extreme-point solution is a basic feasible solution of the linear constraint set: $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$.

Overview

Topics

(i) Unique optimal solution, alternative optimal solutions, and unbounded objective.
(ii) If a finite optimal solution exists, then an extreme-point optimal solution exists.
(iii) Each extreme-point solution is a basic feasible solution of the linear constraint set: $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$.

Basic idea of the simplex algorithm

Iteratively moving from one extreme point to an *adjacent* extreme point, until an extreme point with an optimal solution. How to choose the first, next, and last (optimal) extreme point?

Overview

Topics

(i) Unique optimal solution, alternative optimal solutions, and unbounded objective.

(ii) If a finite optimal solution exists, then an extreme-point optimal solution exists.

(iii) Each extreme-point solution is a basic feasible solution of the linear constraint set: $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$.

Basic idea of the simplex algorithm

Iteratively moving from one extreme point to an *adjacent* extreme point, until an extreme point with an optimal solution. How to choose the first, next, and last (optimal) extreme point? First point (Two phase method).

Representing z and $\mathbf{x}$

The standard linear programming problem:

$$\text{(LP) maximize } z = \mathbf{cx} \text{ subject to } \mathbf{A} \cdot \mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq 0.$$

$\mathbf{A}$ can be partitioned into $\mathbf{A} = (\mathbf{B} : \mathbf{N})$ (4.1)

Based on $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$, we have:

$$\mathbf{B} \cdot \mathbf{x}_B + \mathbf{N} \cdot \mathbf{x}_N = \mathbf{b} \quad (4.2)$$
$$\mathbf{x}_B + \mathbf{B}^{-1} \cdot \mathbf{N} \cdot \mathbf{x}_N = \mathbf{B}^{-1} \cdot \mathbf{b} \quad (4.3)$$
$$\mathbf{x}_B = \mathbf{B}^{-1} \cdot \mathbf{b} - \mathbf{B}^{-1} \cdot \mathbf{N} \cdot \mathbf{x}_N \quad (4.4)$$

The *basic solution is:*

$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_n \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix}$. IF $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$, then $\mathbf{x}$ is a *basic feasible solution*.

The objective function $z = \mathbf{cx}$ can be written as:

$$z = \mathbf{c}_B \mathbf{x}_B + \mathbf{c}_N \mathbf{x}_N \quad (4.5)$$

Representing z and $\mathbf{x}$ cont.

Based on (4.4), $\mathbf{x}_B$ can be described using $\mathbf{x}_N$:

$$z = \mathbf{c}_B(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N) + \mathbf{c}_N\mathbf{x}_N \quad (4.6)$$
$$z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} - (\mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N)\mathbf{x}_N \quad (4.7)$$

A canonical form of z and $\mathbf{x_B}$:

$$z = \mathbf{c}_B(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N) + \mathbf{c}_N\mathbf{x}_N \quad (4.8)$$
$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \quad (4.9)$$

The current basic feasible solution and the objective function z are:

$$z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \quad (4.10)$$
$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix} \geq \mathbf{0} \quad (4.11)$$

Representing z and $\mathbf{x}$ cont.

Letting J denote the index set of the nonbasic variables. The canonical form (4.8-4.9) can be written as:

$$z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} - \sum_{j\in J}(\mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j)x_j \quad (4.12)$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \sum_{j\in J}(\mathbf{B}^{-1}\mathbf{a_j})x_j \quad (4.13)$$

The *key idea* of the simplex method is to move from an extreme point to an improving adjacent extreme point by interchanging a column in **B** and a column in **N**.

Checking for optimality

Based on (4.12) the *rate of change* of z with respect to the nonbasic variable x_j is:

$$\frac{\partial z}{\partial x_j} = -(\mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j - c_j) \qquad (4.14)$$

Thus, if $\frac{\partial z}{\partial x_j} > 0$, then increasing x_j will increase z. $(\mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j - c_j)$ is sometimes referred to as *reduced cost* and is denoted by $(z_j - c_j)$.

A basic feasible solution (4.11) is optimal to (LP) if

$$\frac{\partial z}{\partial x_j} = -(z_j - c_j) = -(\mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j - c_j) \leq 0, \text{ for all } j \in J$$

or, equivalently, if $z_j - c_j = (\mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j - c_j) \geq 0$, for all $j = 1 \ldots n$

$z_j - c_j = 0$ for all basic variables. We can see that $\mathbf{B}^{-1} \mathbf{a}_j = (0, 0, \cdots 1, 0 \cdots 0)^T$; i.e., the only non-zero entry in $\mathbf{B}^{-1} \mathbf{a}_j$ is the j^{th} entry, which is 1. Therefore, $\mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j - c_j = c_j - c_j = 0$.
Meanwhile, based on (4.12), we can directly see that $\frac{\partial z}{\partial x_j} = 0$ for all basic variables.
If $z_j - c_j > 0$ for all $j \in J$, then the current basic solution is the *unique* optimal solution. Otherwise, if $z_j - c_j = 0$ for some nonbasic variable x_j, then there are *alternative optimal solutions*.

Determining the entering and departing variables

A current non-basic variable x_j, if $\frac{\partial z}{\partial x_j}$ is positive, and $\frac{\partial z}{\partial x_j}$ is maximal, then x_j is chosen as the *entering variable*.
This process of choosing the entering variable is called the *steepest-ascent rule*.
However, if we choose another x_j such that $\frac{\partial z}{\partial x_j}$ is positive but not maximal, the simplex method will also work.

x_j will become a basic variable, and some current basic variable x_k will become non-basic. x_k is called the *departing variable*.

The departing variable x_k must satisfy two requirements:

- The columns of $\mathbf{B}$, after $\mathbf{a_k}$ is removed and $\mathbf{a_j}$ is added, can form a basis, i.e. they are linearly independent.
- In order to make x_k non-negative when x_j is increased, x_j needs to satisfy the most restrict upper bound (*minimum ratio test*).

Determining the departing variable: to form a new basis

Theorem (4.1)

Let $\mathbf{B} = (\mathbf{b}_1,\ \mathbf{b}_2, ...,\ \mathbf{b}_m)$ *be a basis for* E^m*, and let* $\mathbf{a} \in E^m$*,* $\mathbf{a} \neq \mathbf{0}$*. Then* $\mathbf{a}$ *can be written uniquely as a linear combination of* $\mathbf{b}_1,\ \mathbf{b}_2, ...,\ \mathbf{b}_m$*.*

Proof.

Because $\mathbf{B} = (\mathbf{b}_1,\ \mathbf{b}_2, ...,\ \mathbf{b}_m)$ is a basis for E^m, then $\mathbf{a}$ can be written as a linear combination of $(\mathbf{b}_1,\ \mathbf{b}_2, ...,\ \mathbf{b}_m)$. We need to show that this linear combination is unique. Suppose $\mathbf{a}$ can be represented as two different linear combinations of $\mathbf{B}$:

$$\mathbf{a} = \sum_{j=1}^{m} \lambda_j \mathbf{b}_j, \text{where } \lambda_j \in E^1, \text{ for all } j = 1, \ldots, m$$

Determining the departing variable: to form a new basis

Theorem (4.1)

Let $\mathbf{B} = (\mathbf{b}_1, \ \mathbf{b}_2, ..., \ \mathbf{b}_m)$ *be a basis for* E^m*, and let* $\mathbf{a} \in E^m$*,* $\mathbf{a} \neq \mathbf{0}$*. Then* $\mathbf{a}$ *can be written uniquely as a linear combination of* $\mathbf{b}_1, \ \mathbf{b}_2, ..., \ \mathbf{b}_m$*.*

Proof.

Because $\mathbf{B} = (\mathbf{b}_1, \ \mathbf{b}_2, ..., \ \mathbf{b}_m)$ is a basis for E^m, then $\mathbf{a}$ can be written as a linear combination of $(\mathbf{b}_1, \ \mathbf{b}_2, ..., \ \mathbf{b}_m)$. We need to show that this linear combination is unique. Suppose $\mathbf{a}$ can be represented as two different linear combinations of $\mathbf{B}$:

$$\mathbf{a} = \sum_{j=1}^{m} \lambda_j \mathbf{b}_j, \text{where } \lambda_j \in E^1, \text{ for all } j = 1, \ldots, m \qquad (4.15)$$

$$\mathbf{a} = \sum_{j=1}^{m} \mu_j \mathbf{b}_j, \text{where } \mu_j \in E^1, \text{ for all } j = 1, \ldots, m$$

Determining the departing variable: to form a new basis

Theorem (4.1)

Let $\mathbf{B} = (\mathbf{b}_1,\ \mathbf{b}_2, ...,\ \mathbf{b}_m)$ *be a basis for* E^m*, and let* $\mathbf{a} \in E^m$*,* $\mathbf{a} \neq \mathbf{0}$*. Then* $\mathbf{a}$ *can be written uniquely as a linear combination of* $\mathbf{b}_1,\ \mathbf{b}_2, ...,\ \mathbf{b}_m$*.*

Proof.

Because $\mathbf{B} = (\mathbf{b}_1,\ \mathbf{b}_2, ...,\ \mathbf{b}_m)$ is a basis for E^m, then $\mathbf{a}$ can be written as a linear combination of $(\mathbf{b}_1,\ \mathbf{b}_2, ...,\ \mathbf{b}_m)$. We need to show that this linear combination is unique. Suppose $\mathbf{a}$ can be represented as two different linear combinations of $\mathbf{B}$:

$$\mathbf{a} = \sum_{j=1}^{m} \lambda_j \mathbf{b}_j, \text{where } \lambda_j \in E^1,\ \text{for all } j = 1, \ldots, m \qquad (4.15)$$

$$\mathbf{a} = \sum_{j=1}^{m} \mu_j \mathbf{b}_j, \text{where } \mu_j \in E^1,\ \text{for all } j = 1, \ldots, m \qquad (4.16)$$

Subtracting (4.16) from(4.15) yields $\mathbf{0} = \sum_{j=1}^{m} (\lambda_j - \mu_j)_{\mathbf{j}}$, which represents $\mathbf{0}$ as a linear combination of columns in $\mathbf{B}$ where some coefficient is not 0, which is impossible since the columns in $\mathbf{B}$ are linearly independent. So we know that $\mathbf{a}$ can be written uniquely as a linear combination of $\mathbf{b}_1,\ \mathbf{b}_2, ...,\ \mathbf{b}_m$. □

Another theorem

Theorem (4.2)

Let $\mathbf{B} = (\mathbf{b}_1,\ \mathbf{b}_2, ...,\ \mathbf{b}_m)$ *be a basis for* E^m*, and let* $\mathbf{a} \in E^m$*,* $\mathbf{a} \neq \mathbf{0}$ *be represented by* $\mathbf{a} = \sum_{j=1}^{m} \lambda_j b_j$*. Without loss of generality, suppose* $\lambda_m \neq 0$*. Then* $\mathbf{b}_1,\ \mathbf{b}_2, ...,\ \mathbf{b}_{m-1},\ \mathbf{a}$ *form a basis for* E^m*.*

Proof.

We simply need to show that $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{m-1}, \mathbf{a}$ are linearly independent. By contradiction, suppose they are not. There exist $\gamma_1,\ \gamma_2,\ \ldots\ \gamma_{m-1},\ \delta \in E^1$, which are not all zero such that

$$\sum_{j=1}^{m-1} \gamma_j \mathbf{b}_j + \delta \mathbf{a} = \mathbf{0} \qquad (4.19).$$

If $\delta = 0$, then some γ_j, for $1 \leq j \leq m-1$, is non-zero, and $\sum_{j=1}^{m-1} \gamma_j \mathbf{b}_j = \mathbf{0}$. It is impossible since $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{m-1}$ are linearly independent. So we know that $\delta = 0$. We have

$$\mathbf{a} = \sum_{j=1}^{m} \lambda_j \mathbf{b}_j \qquad (4.20)$$

Proof continues . . . □

Determining the departing variable, cont.

Proof of Theorem 4.2 cont.

Substituting (4.20) into (4.19) yields

$$\sum_{j=1}^{m-1} \gamma_j \mathbf{b}_j + \delta \sum_{j=1}^{m} \lambda_j \mathbf{b}_j = \sum_{j=1}^{m-1} (\gamma_j + \delta\lambda_j)\mathbf{b}_j + \delta\lambda_m \mathbf{b}_m = \mathbf{0} \qquad (4.21)$$

Because $\delta \neq 0$ and $\lambda_m \neq 0$, which is assumed by this theorem, it follows that $\delta\lambda_m \neq 0$, and (4.21) contradicts the fact that $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m$ are linearly independent.
Thus $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_{m-1}, \mathbf{a}$ are linearly independent and form a basis for E^m. □

Define for each $\mathbf{a}_j$ associated with a non-basic variable x_j:

$$\boldsymbol{\alpha}_j = \mathbf{B}^{-1}\mathbf{a}_j \qquad (4.22)$$

Multiply the both sides of (4.22) by B yields

$$\mathbf{a}_j = \mathbf{B}\boldsymbol{\alpha}_j = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m) \begin{pmatrix} \boldsymbol{\alpha}_{1,j} \\ \boldsymbol{\alpha}_{2,j} \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{\alpha}_{m,j} \end{pmatrix} = \sum_{i=1}^{m} \boldsymbol{\alpha}_{i,j}\mathbf{b}_j \qquad (4.23)$$

So $\mathbf{a}_j$ is described as a (unique) linear combination of columns in $\mathbf{B}$.
$\mathbf{a}_j$ can be exchanged with any column $\mathbf{b}_i$ of $\mathbf{B}$ for which $\alpha_{i,j} \neq 0$ for form a new basis.

Determining the departing variable: minimum ratio test

From (4.13), we see the rate of change of $\mathbf{x}_B$ w.r.t. x_j, which is the entering variable

$$\frac{\partial \mathbf{x}_B}{\partial x_j} = -\mathbf{B}^{-1}\mathbf{a}_j = -\boldsymbol{\alpha}_j \qquad (4.24)$$

$\mathbf{x}_B$ will change according to a_j, while $\mathbf{x}_B$ is non-negative:

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} + x_j(\mathbf{B}^{-1}\mathbf{a}_j) = \mathbf{B}^{-1}\mathbf{b} - x_j\boldsymbol{\alpha}_j \qquad (4.25)$$
$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - x_j\boldsymbol{\alpha}_j \geq \mathbf{0} \qquad (4.26)$$

Now let

$$\mathbf{B}^{-1}\mathbf{b} = \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_m \end{pmatrix} \qquad (4.27)$$

Determining the departing variable: minimum ratio test, cont.

From (4.26) (4.27)

$$\begin{pmatrix}\beta_1\\ \beta_2\\ \cdot\\ \cdot\\ \cdot\\ \beta_m\end{pmatrix} - x_j \begin{pmatrix}\boldsymbol{\alpha}_{1,j}\\ \boldsymbol{\alpha}_{2,j}\\ \cdot\\ \cdot\\ \cdot\\ \boldsymbol{\alpha}_{m,j}\end{pmatrix} \geq 0 \qquad (4.28)$$

An upper bound on x_j can be found as

$$x_j \leq \text{minimum}\left\{\frac{\beta_i}{\boldsymbol{\alpha}_{i,j}} : \boldsymbol{\alpha}_{i,j} > 0\right\} \qquad (4.29)$$

- The basic variable x_k that causing the minimum upper bound of x_j (when $x_k = 0$) is the *departing variable*.
- x_k surely satisfy the requirement that after x_k and x_j are exchanged, B should still be a basis, because $\boldsymbol{\alpha}_{k,j} > 0$.

Optimality conditions and directions

Let the current basic feasible solution be:

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix} \geq \mathbf{0} \qquad (4.30)$$

When x_j increases, as shown by (4.13), $\mathbf{x}_B$ changes according to:

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} + x_j(-\mathbf{B}^{-1}\mathbf{a}_j) = \mathbf{B}^{-1}\mathbf{b} + x_j(-\alpha_j) \qquad (4.31)$$

When x_j increases, $\mathbf{x}_N$ changes according to:

$$\mathbf{x}_N = \mathbf{0} + x_j\mathbf{e}_j = \mathbf{0} + x_j \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \\ . \\ . \\ . \\ 0 \end{pmatrix} \qquad (4.32)$$

where 1 in $\mathbf{e}_j$ in the j^{th} position.

Optimality conditions and directions, cont.

As x_j is increased, the solution $\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix}$ is moving in the direction

$$\mathbf{d} = \begin{pmatrix} -\boldsymbol{\alpha}_j \\ \mathbf{e}_j \end{pmatrix} \qquad (4.33)$$

Now consider cd, where c is the gradient of the objective function

$$\begin{aligned} \mathbf{cd} &= (\mathbf{c}_B \mathbf{c}_N) \begin{pmatrix} -\boldsymbol{\alpha}_j \\ \mathbf{e}_j \end{pmatrix} \\ &= \mathbf{c}_B \boldsymbol{\alpha}_j + \mathbf{c}_N \mathbf{e}_j = -\mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j + c_j = -z_j + c_j = -(z_j - c_j) \end{aligned} \qquad (4.34)$$

- $-(z_j - c_j) > 0$ since that is the way x_j is chosen.
- $\mathbf{cd} = -(z_j - c_j) > 0$ implies that the angle between **c** and **d** is acute.
- It also proves that the new basic solution has a larger objective value than the previous basic solution. (moving to an extreme point closer to optimal).

Checking for an unbounded objective

When examining the minimum ratio test on α_j, if we find that $\alpha_{i,j} \leq 0$, for all i, then x_j can be increased indefinitely.

The objective function can be increased indefinitely with x_j by the points (solutions) defined as follows:

$$\begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix} + x_j \begin{pmatrix} -\alpha_j \\ \mathbf{e}_j \end{pmatrix} \qquad (4.35)$$

Note that these points form a ray whose end point is $\begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix}$.

Example, algebra of the Simplex method.

maximize $z = 2x_1 + 3x_2$

subject to

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 4 \\ 2x_1 + x_2 + x_4 &= 18 \\ x_2 + x_5 &= 10 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

This problem can be summarized as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 4 \\ 18 \\ 10 \end{pmatrix}$$

$$\mathbf{c} = \begin{pmatrix} 2 & 3 & 0 & 0 & 0 \end{pmatrix}$$

Example, cont. starting Simplex method

Always chose the starting basis matrix $\mathbf{B} = \mathbf{I}$

$$\mathbf{B} = (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

$$\mathbf{x}_B = \begin{pmatrix} x_{B,1} \\ x_{B,2} \\ x_{B,3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Solving z and $\mathbf{x}_B$ in terms of x_N:

$$
\begin{aligned}
z &= 2x_1 + 3x_2\\
x_3 &= 4 - x_1 + x_2\\
x_4 &= 18 - 2x_1 - x_2\\
x_3 &= 10 - x_2
\end{aligned}
$$

Starting solution is obtained by setting the nonbasic variables equal to zero

$$
\begin{aligned}
z &= 0\\
\mathbf{x}_B &= \begin{pmatrix} x_{B,1}\\ x_{B,2}\\ x_{B,3}\end{pmatrix} = \begin{pmatrix} x_3\\ x_4\\ x_5\end{pmatrix} = \begin{pmatrix} 4\\ 18\\ 10\end{pmatrix}\\
\mathbf{x}_N &= \begin{pmatrix} 0\\ 0\end{pmatrix}
\end{aligned}
$$

Choosing the entering variable

$\partial z/\partial x_2 = 3$ (maximal).
$\partial z/\partial x_1 = 2$.
We choose x_2 as the entering variable. By the way, the current basic solution is not optimal.

Choosing the departing variables

Entering x_2

As x_2 is increased, we must ensure that x_3 and x_4 and x_5 remain nonnegative. By (4.25)(4.26),

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} + x_2(-\mathbf{B}^{-1}\mathbf{a}_2) = \begin{pmatrix} 4 \\ 18 \\ 10 \end{pmatrix} - x_2 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \geq 0$$

x_2 needs to satisfy the most restrictive upper bound $x_2 \leq 10$ due to x_5.
x_5 is the *departing variable*.

Pivot

The new canonically representation of z and $\mathbf{x}_B$ is are formed using $x_2 = 10 - x_5$ to eliminate x_2; i.e., to represent the basic variables x_2, x_3 and x_4 by the non-basic variables x_1 and x_5.

$$\begin{aligned} z = &\quad 2x_1 + 3(10 - x_5) = 30 + 2x_1 - 3x_5 \\ x_3 = &\quad 4 - x_1 + 2(10 - x_5) = 24 - x_1 - 2x_5 \\ x_4 = &\quad 18 - 2x_1 - (10 - x_5) = 8 - 2x_1 + x_5 \\ x_2 = &\quad 10 - x_5 \end{aligned}$$

The new current basic solution and basis matrix:

$$
\begin{aligned}
z &= 30 \\
\mathbf{x}_B &= \begin{pmatrix} x_{B,1} \\ x_{B_2} \\ x_{B_3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_2 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \\ 10 \end{pmatrix} \\
\mathbf{x}_N &= \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\mathbf{B} &= (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_2) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$