NALITY MATCHING PROBLEM, described in Section 10.5, and its weighted version which we postpone to Sections 11.2 and 11.3.

10.1 Bipartite Matching

Since the CARDINALITY MATCHING PROBLEM is easier if G is bipartite, we shall deal with this case first. In this section, a bipartite graph G is always assumed to have the bipartition $V(G) = A \dot{\cup} B$. Since we may assume that G is connected, we can regard this bipartition as unique (Exercise 19 of Chapter 2).

For a graph G, let $\nu(G)$ denote the maximum cardinality of a matching in G, while $\tau(G)$ is the minimum cardinality of a vertex cover in G.

Theorem 10.2. (König [1931]) *If G is bipartite, then $\nu(G) = \tau(G)$.*

Proof: Consider the graph $G' = (V(G) \dot{\cup} \{s,t\},\ E(G) \cup \{\{s,a\} : a \in A\} \cup \{\{b,t\} : b \in B\})$. Then $\nu(G)$ is the maximum number of vertex-disjoint s-t-paths, while $\tau(G)$ is the minimum number of vertices whose deletion makes t unreachable from s. The theorem now immediately follows from Menger's Theorem 8.10. □

$\nu(G) \leq \tau(G)$ evidently holds for any graph (bipartite or not), but we do not have equality in general (as the triangle K_3 shows).

Several statements are equivalent to König's Theorem. Hall's Theorem is probably the best-known version.

Theorem 10.3. (Hall [1935]) *Let G be a bipartite graph with bipartition $V(G) = A \dot{\cup} B$. Then G has a matching covering A if and only if*

$$|\Gamma(X)| \geq |X| \qquad \textit{for all } X \subseteq A. \tag{10.1}$$

Proof: The necessity of the condition is obvious. To prove the sufficiency, assume that G has no matching covering A, i.e. $\nu(G) < |A|$. By Theorem 10.2 this implies $\tau(G) < |A|$.

Let $A' \subseteq A$, $B' \subseteq B$ such that $A' \cup B'$ covers all the edges and $|A' \cup B'| < |A|$. Obviously $\Gamma(A \setminus A') \subseteq B'$. Therefore $|\Gamma(A \setminus A')| \leq |B'| < |A| - |A'| = |A \setminus A'|$, and the Hall condition (10.1) is violated. □

It is worthwhile to mention that it is not too difficult to prove Hall's Theorem directly. The following proof is due to Halmos and Vaughan [1950]:
Second Proof of Theorem 10.3: We show that any G satisfying the Hall condition (10.1) has a matching covering A. We use induction on $|A|$, the cases $|A| = 0$ and $|A| = 1$ being trivial.

If $|A| \geq 2$, we consider two cases: If $|\Gamma(X)| > |X|$ for every nonempty proper subset X of A, then we take any edge $\{a,b\}$ ($a \in A$, $b \in B$), delete its two vertices and apply induction. The smaller graph satisfies the Hall condition because $|\Gamma(X)| - |X|$ can have decreased by at most one for any $X \subseteq A \setminus \{a\}$.