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affine combination

Definition

Let V be a vector space over a division ring D. An affine combination of a finite set of vectors $v_1, \ldots, v_n \in V$ is a linear combination of the vectors

$$k_1v_1 + \cdots + k_nv_n$$

such that $k_i \in D$ subject to the condition $k_1 + \cdots + k_n = 1$. In effect, an affine combination is a weighted average of the vectors in question.

For example, $v = \frac{1}{2}v_1 + \frac{1}{2}v_2$ is an affine combination of v_1 and v_2 provided that the characteristic of *D* is not 2. *v* is known as the midpoint of v_1 and v_2 . More generally, if char(*D*) does not divide *m*, then

$$v = \frac{1}{m}(v_1 + \dots + v_m)$$

is an affine combination of the v_i 's. v is the barycenter of v_1, \ldots, v_n .

Relations with Affine Subspaces

Assume now char(D) = 0. Given $v_1, \ldots, v_n \in V$, we can form the set A of all affine combinations of the v_i 's. We have the following

A is a finite dimensional affine subspace. Conversely, a finite dimensional affine subspace A is the set of all affine combinations of a finite set of vectors in A.

Proof. Suppose A is the set of affine combinations of v_1, \ldots, v_n . If n = 1, then A is a singleton $\{v\}$, so A = 0 + v, where 0 is the null subspace of V. If n > 1, we may pick a non-zero vector $v \in A$. Define $S = \{a - v \mid a \in A\}$. Then for any $s \in S$ and $d \in D$, ds = d(a - v) = da + (1 - d)v - v. Since $da + (1 - d)v \in A$, $ds \in S$. If $s_1, s_2 \in S$, then $\frac{1}{2}(s_1 + s_2) = \frac{1}{2}((a_1 - v) + (a_2 - v)) = \frac{1}{2}(a_1 + a_2) - v \in S$, since $\frac{1}{2}(a_1 + a_2) \in A$. So $\frac{1}{2}(s_1 + s_2) \in S$. Therefore, $s_1 + s_2 = 2(\frac{1}{2}(s_1 + s_2)) \in S$. This shows that S is a vector subspace of V and that A = S + v is an affine subspace.

Conversely, let A be a finite dimensional affine subspace. Write A = S + v, where S is a subspace of V. Since $\dim(S) = \dim(A) = n$, S has a basis $\{s_1, \ldots, s_n\}$. For each $i = 1, \ldots, n$, define $v_i = ns_i + v$. Given $a \in A$, we have

$$a = s + v = k_1 s_1 + \dots + k_n s_n + v$$

= $\frac{k_1}{n} (v_1 - v) + \dots + \frac{k_n}{n} (v_n - v) + v$
= $\frac{k_1}{n} v_1 + \dots + \frac{k_n}{n} v_n + (1 - \frac{k_1}{n} - \dots - \frac{k_n}{n}) v$

From this calculation, it is evident that *a* is an affine combination of v_1, \ldots, v_n , and v.

When *A* is the set of affine combinations of two distinct vectors v, w, we see that *A* is a line, in the sense that A = S + v, a translate of a one-dimensional subspace *S* (a line through 0). Every element in *A* has the form

dv + (1 - d)w, $d \in D$. Inspecting the first part of the proof in the previous proposition, we see that the argument involves no more than two vectors at a time, so the following useful corollary is apparant:

A is an affine subspace iff for every pair of vectors in A , the line formed by the pair is also in A .

Note, however, that the A in the above corollary is not assumed to be finite dimensional.

Remarks.

If one of v₁,..., v_n is the zero vector, then A coincides with S. In other words, an affine subspace is a vector subspace if it contains the zero vector.

• Given
$$A = \{k_1v_1 + \dots + k_nv_n \mid v_i \in V, k_i \in D, \sum k_i = 1\}$$
, the subset

$$\{k_1v_1 + \dots + k_nv_n \in A \mid k_i = 0\}$$

is also an affine subspace.

Affine Independence

Since every element in a finite dimensional affine subspace A is an affine combination of a finite set of vectors in A, we have the similar concept of a spanning set of an affine subspace. A minimal spanning set M of an affine subspace is said to be affinely independent. We have the following three equivalent characterization of an affinely independent subset M of a finite dimensional affine subspace:

- 1. $M = \{v_1, \ldots, v_n\}$ is affinely independent.
- 2. every element in A can be written as an affine combination of elements in M in a unique fashion.
- 3. for every $v \in M$, $N = \{v_i v \mid v \neq v_i\}$ is linearly independent.

Proof. We will proceed as follows: (1) implies (2) implies (3) implies (1).

(1) implies (2). If $a \in A$ has two distinct representations $k_1v_1 + \cdots + k_nv_n = a = r_1v_1 + \cdots + r_nv_n$, we may assume, say $k_1 \neq r_1$. So $k_1 - r_1$ is invertible with inverse $t \in D$. Then

$$v_1 = t(r_2 - k_2)v_2 + \dots + t(r_n - k_n)v_n.$$

Furthermore,

$$\sum_{i=2}^{n} t(r_i - k_i) = t(\sum_{i=2}^{n} r_i - \sum_{i=2}^{n} k_i) = t(1 - r_1 - 1 + k_1) = 1.$$

So for any $b \in A$, we have

$$b = s_1 v_1 + \dots + s_n v_n = s_1 (t(r_2 - k_2)v_2 + \dots + t(r_n - k_n)v_n) + \dots + s_n v_n.$$

The sum of the coefficients is easily seen to be 1, which implies that $\{v_2, \ldots, v_n\}$ is a spanning set of A that is smaller than M, a contradiction.

(2) implies (3). Pick $v = v_1$. Suppose $0 = s_2(v_2 - v_1) + \dots + s_n(v_n - v_1)$. Expand and we have $0 = (-s_2 - \dots - s_n)v_1 + s_2v_2 + \dots + s_nv_n$. So $(1 - s_2 - \dots - s_n)v_1 + s_2v_2 + \dots + s_nv_n = v_1 \in A$. By assumption, there is exactly one way to express v_1 , so we conclude that $s_2 = \dots = s_n = 0$.

(3) implies (1). If M were not minimal, then some $v \in M$ could be expressed as an affine combination of the remaining vectors in M. So suppose $v_1 = k_2v_2 + \cdots + k_nv_n$. Since $\sum k_i = 1$, we can rewrite this as $0 = k_2(v_2 - v_1) + \cdots + k_n(v_n - v_1)$. Since not all $k_i = 0$, $N = \{v_2 - v_1, \dots, v_n - v_1\}$ is not linearly independent. \Box

Remarks.

- If $\{v_1, \ldots, v_n\}$ is affinely independent set spanning A, then dim(A) = n 1.
- More generally, a set M (not necessarily finite) of vectors is said to be affinely independent if there is a vector $v \in M$, such that $N = \{w v \mid v \neq w \in M\}$ is linearly independent (every finite subset of N is linearly independent). The above three characterizations are still valid in this general setting. However, one must be careful that an affine combination is a finitary operation so that when we take the sum of an infinite number of vectors, we have to realize that only a finite number of them are non-zero.
- Given any set S of vectors, the affine hull of S is the smallest affine subspace A that contains every vector of S ,

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