8 The Two-Phase Simplex Method

The LP we solved in the previous lecture allowed us to find an initial BFS very easily. In cases where such an obvious candidate for an initial BFS does not exist, we use an additional phase I to find a BFS. In phase II we then proceed as in the previous lecture.

Consider the LP to

$$\begin{aligned}
\text{maximize} \quad & -6x_1 - 3x_2 \\
\text{subject to} \quad & x_1 + x_2 \geqslant 1 \\
& 2x_1 - x_2 \geqslant 1 \\
& 3x_2 \leqslant 2 \\
& x_1, x_2 \geqslant 0,
\end{aligned}$$

and introduce slack variables to obtain

$$\begin{aligned}
\text{maximize} \quad & -6x_1 - 3x_2 \\
\text{subject to} \quad & x_1 + x_2 - z_1 = 1 \\
& 2x_1 - x_2 - z_2 = 1 \\
& 3x_2 + z_3 = 2 \\
& x_1, x_2, z_1, z_2, z_3 \geqslant 0.
\end{aligned}$$

Unfortunately, the basic solution with $x_1 = x_2 = 0$, $z_1 = z_2 = -1$, and $z_3 = 2$ is not feasible. We can, however, add an *artificial variable* to the left-hand side of each constraint where the slack variable and the right-hand side have opposite signs, and then minimize the sum of the artificial variables starting from the obvious BFS where the artificial variables are non-zero instead of the corresponding slack variables. In the example, we

$$\begin{aligned}
\text{minimize} \quad & y_1 + y_2 \\
\text{subject to} \quad & x_1 + x_2 - z_1 + y_1 = 1 \\
& 2x_1 - x_2 - z_2 + y_2 = 1 \\
& 3x_2 + z_3 = 2 \\
& x_1, x_2, z_1, z_2, z_3, y_1, y_2 \geqslant 0,
\end{aligned}$$

and the goal of phase I is to solve this LP starting from the BFS where $x_1 = x_2 = z_1 = z_2 = 0$, $y_1 = y_2 = 1$, and $z_3 = 2$. If the original problem is feasible, we will be able to find a BFS where $y_1 = y_2 = 0$. This automatically gives us an initial BFS for the original problem.

In summary, the two-phase simplex method proceeds as follows:

1. Bring the constraints into equality form. For each constraint in which the slack variable and the right-hand side have opposite signs, or in which there is no slack

variable, add a new artificial variable that has the same sign as the right-hand side.

2. Phase I: minimize the sum of the artificial variables, starting from the BFS where the absolute value of the artificial variable for each constraint, or of the slack variable in case there is no artificial variable, is equal to that of the right-hand side.
3. If some artificial variable has a positive value in the optimal solution, the original problem is infeasible; stop.
4. Phase II: solve the original problem, starting from the BFS found in phase I.

While the original objective is not needed for phase I, it is useful to carry it along as an extra row in the tableau, because it will then be in the appropriate form at the beginning of phase II. In the example, phase I therefore starts with the following tableau:

	x_1	x_2	z_1	z_2	z_3	y_1	y_2	
y_1	1	1	−1	0	0	1	0	1
y_2	2	−1	0	−1	0	0	1	1
z_3	0	3	0	0	1	0	0	2
II	−6	−3	0	0	0	0	0	0
I	3	0	−1	−1	0	0	0	2

Note that the objective for phase I is written in terms of the variables that are *not* in the basis. This can be obtained by first writing it in terms of y_1 and y_2, such that we have -1 in the columns for y_1 and y_2 and 0 in all other columns because we are *maximizing* $-y_1 - y_2$, and then adding the first and second row to make the entries for all variables in the basis equal to zero.

Phase I now proceeds by pivoting on a_{21} to get

	x_1	x_2	z_1	z_2	z_3	y_1	y_2	
	0	$\frac{3}{2}$	−1	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	$\frac{1}{2}$
	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$
	0	3	0	0	1	0	0	2
II	0	−6	0	−3	0	0	3	3
I	0	$\frac{3}{2}$	−1	$\frac{1}{2}$	0	0	$-\frac{3}{2}$	$\frac{1}{2}$

and on a_{14} to get

	x_1	x_2	z_1	z_2	z_3	y_1	y_2	
	0	3	−2	1	0	2	−1	1
	1	1	−1	0	0	1	0	1
	0	3	0	0	1	0	0	2
II	0	3	−6	0	0	6	0	6
I	0	0	0	0	0	−1	−1	0