Combinatorial Optimization - Homework I

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1 Instructions

1. The homework is due on February 7, in class.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.
4. The work must be entirely your own. You are expressly **prohibited** from consulting with colleagues or the internet (with the exception of the material on the course website).

2 Problems

1. Consider the feasibility version of the linear programming problem, in which you are given a matrix $\mathbf{A}_{m \times n}$, a vector $\mathbf{b}_{m \times 1}$ and asked, if there exists $\mathbf{x} \in \Re^n$, such that

   $$\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \tag{1}$$

   In the integer programming counterpart, we are required to find an integral $\mathbf{x}$, which satisfies System (1). Show that an oracle for the integer programming problem can be used to solve the linear programming problem in polynomial time.

2. Is the Fourier-Motzkin elimination procedure polynomially convergent? If yes, provide a proof; if no, provide a counterexample.

3. In class, I showed that the same combinatorial optimization problem can have more than one integer programming formulation. Let F and G denote two formulations for a problem P. Formulation F is said to be stronger than formulation G, if the linear programming relaxation of F is a strict subset of the linear programming relaxation of G.

   Consider the following two integer programming formulations of a problem:

$$\begin{aligned}
\mathbf{F} : 2 \cdot x_1 + 2 \cdot x_2 + x_3 + x_4 &\leq 2 \\
x_1 &\leq 1 \\
x_2 &\leq 1 \\
x_3 &\leq 1 \\
x_4 &\leq 1 \\
x_i &\geq 0, \; i = 1, 2, 3, 4
\end{aligned}$$

$$
\begin{aligned}
\mathbf{G}: \; x_1 + x_2 + x_3 &\leq 1 \\
x_1 + x_2 + x_4 &\leq 1 \\
x_i &\geq 0, \; i = 1, 2, 3, 4
\end{aligned}
$$

Which one is stronger?

4. Let $\mathbf{A}$ denote an $m \times n$ matrix and $\mathbf{b}$ denote an $m \times 1$ vector. Using the Strong Duality Theorem, prove that that either

$$\exists \mathbf{x} \in \Re^n_+ \;\; \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

or (mutually exclusively)

$$
\begin{aligned}
\exists \mathbf{y} \in \Re^m_+ \;\; \mathbf{y} \cdot \mathbf{A} &\geq \mathbf{0} \\
\mathbf{y} \cdot \mathbf{b} &< 0.
\end{aligned}
$$

5. Let $x_1, x_2, \ldots x_k$ be points in $\Re^n$. Argue that the following statements are equivalent:

   (i) $x_1, x_2, \ldots x_k$ are affinely independent.

   (ii) $(x_2 - x_1), (x_3 - x_1), \ldots (x_k - x_1)$ are linearly independent.

   (iii) $(x_1, 1), (x_2, 1), \ldots (x_k, 1)$ are linearly independent.