Combinatorial Optimization - Homework II

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1 Instructions

1. The homework is due on March 7, in class.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.
4. Some of the problems require at least a passing familiarity with the Tableau method for implementing the simplex algorithm.
5. The work must be entirely your own. You are expressly **prohibited** from consulting with colleagues or the internet (with the exception of the material on the course website).

2 Problems

1. Let $\mathbf{S}$ denote a convex set and let $\mathbf{x}$ denote a point in $\mathbf{S}$. Argue that $\mathbf{x}$ is an extreme point of $\mathbf{S}$, if and only if $\mathbf{S} - \{\mathbf{x}\}$ is convex.
2. Use the Tableau method to prove the Strong Duality theorem.
3. Use the Tableau method to prove the Strong Optimal-Basis theorem.
4. Prove that if $\mathbf{A}$ is a (Totally Unimodular) TU matrix, then any matrix obtained from $\mathbf{A}$, by performing a Gauss-Jordan pivot is also TU.
5. A polyhedral system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ is said to be **Totally Dual Integral** (TDI), if the minimum in the Linear Programming duality equation
$$\max\{\mathbf{c} \cdot \mathbf{x} \,:\, \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\} = \min\{\mathbf{y} \cdot \mathbf{b} \,:\, \mathbf{y} \cdot \mathbf{A} = \mathbf{c},\ \mathbf{y} \geq \mathbf{0}\}$$
has an integral solution $\mathbf{y}$ for each integral $\mathbf{c}$, for which the minimum is finite.

   Consider the following conjecture:

   Conjecture 2.1 *An integral matrix $\mathbf{A}$ is Totally Unimodular, if and only if the system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$ is TDI, for each vector $\mathbf{b}$.*

   Is the conjecture valid?