Combinatorial Optimization - Homework III

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1 Instructions

1. The homework is due on April 10, in class.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.
4. The work must be entirely your own. You are expressly **prohibited** from consulting with colleagues or the internet (with the exception of the material on the course website).

2 Problems

1. Assume that you are given a flow network: $\mathbf{G} = \langle \mathbf{V}, \mathbf{E}, \mathbf{c}, s, t \rangle$. The notion of min-cuts in flow networks has been discussed in class. The cardinality of a cut in a flow network is defined as the *number* of edges in that cut. Describe a polynomial time algorithm for the problem of finding the smallest cardinality min-cut in a flow network. You may assume that all edge capacities are integral.
2. An undirected bipartite graph $\mathbf{G} = \langle L \cup R, E \rangle$ is said to be **d-regular**, if every vertex has degree exactly d. Argue that every d-regular bipartite graph has a matching of cardinality $|\mathbf{L}|$.
3. Consider the maximum flow problem in a flow network: $\mathbf{G} = \langle \mathbf{V}, \mathbf{E}, \mathbf{c}, s, t \rangle$. Write down the linear program for this problem and its dual. Provide a proof of the Max-Flow/Min-Cut Theorem using linear programming duality and total unimodularity. (You will need to interpret the dual problem).
4. Let $M = \langle (E(M), \mathcal{I}(M) \rangle$ denote matroid. Let $X \in \mathcal{I}(M)$, and $X + e \notin \mathcal{I}(M)$. Argue that $X + e$ contains a *unique* circuit of M. Provide a direct proof of this property for graphic matroids. Also provide a counterexample to show that this property need not hold for a general independence system.
5. Let S be a finite set and let $S_1, S_2, \ldots, S_k$ be a *partition* of S into k nonempty disjoint subsets.
   Let $\mathcal{I} = \{A : |A \cap S_i| \leq 1, \ i = 1, 2, \ldots, k\}$. Argue that the structure $\langle S, \mathcal{I} \rangle$ is a matroid.