Combinatorial Optimization - Homework IV

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1 Instructions

1. The homework is due on May 9.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.
4. The work must be entirely your own. You are expressly **prohibited** from consulting with colleagues or the internet (with the exception of the material on the course website).

2 Problems

1. A subset C of a vector space V is said to be a convex cone, if $\forall x, y \in C$, $\alpha \cdot x + \beta \cdot y \in C$. Prove or disprove the following statements:
   (a) The union of two convex cones in a vector space is a convex cone.
   (b) The intersection of two convex cones in a vector space is a convex cone.
2. Let $\mathbf{P}$ denote a polyhedron and $\mathbf{P_I}$ denote the convex hull of the lattice points in $\mathbf{P}$. Argue that the following statements are equivalent:
   (i) $\mathbf{P} = \mathbf{P_I}$.
   (ii) Each face of $\mathbf{P}$ contains integral vectors.
3. Consider the matrix
$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
   Argue that $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has an integral solution, whenever $\mathbf{b}$ is integral. Is $\mathbf{A}$ Totally Unimodular? Explain.
4. Apply the Gomory cutting plane method to solve the following integer program:
$$\begin{array}{rcl} \max\ 18 \cdot x_1 + 12 \cdot x_2 & & \\ \text{subject to} & & \\ 2 \cdot x_1 - x_2 & \leq & 5 \\ 2 \cdot x_1 + 3 \cdot x_2 & \leq & 13 \\ x_1, x_2 & \in & \mathcal{Z} \end{array}$$

5. Consider the integer program:

$$\begin{aligned} \max\ -x_0 \\ \text{subject to} \\ x_0 + 2 \cdot \sum_{i=1}^{n} x_i &= n \\ x_i &\in \{0,1\},\ i = 1, 2, \ldots, n \end{aligned}$$

Argue that when Branch and Bound is applied to this program, an exponential number of subprograms are generated.