First Order Logic - Satisfiability and Validity

K. Subramani[1]

[1]Lane Department of Computer Science and Electrical Engineering
West Virginia University

6 February, 13 February 2013

Outline

Outline

Outline

Basics

Basics

Definition

Given a FOL formula F, the satisfiability problem is concerned with the following question:

Basics

Definition

Given a FOL formula F, the satisfiability problem is concerned with the following question: Is there some interpretation I, such that $I \models F$?

Basics

Definition

Given a FOL formula F, the satisfiability problem is concerned with the following question: Is there some interpretation I, such that $I \models F$?
Given a FOL formula F, the validity problem is concerned with the following question:

Basics

Definition

Given a FOL formula F, the satisfiability problem is concerned with the following question: Is there some interpretation I, such that $I \models F$?
Given a FOL formula F, the validity problem is concerned with the following question: Is it the case that for all interpretations I, $I \models F$?

Motivation

Motivation

Example

From “All humans are mortal”,

Motivation

Example

From “All humans are mortal”, and “Socrates is human”,

Motivation

Example

From “All humans are mortal”, and “Socrates is human”, we wish to conclude that

Motivation

Example

From “All humans are mortal”, and “Socrates is human”, we wish to conclude that “Socrates is mortal.”

Motivation

Example

From “All humans are mortal”, and “Socrates is human”, we wish to conclude that “Socrates is mortal.” We therefore need rules to reason about predicate expressions.

Motivation

Example

From “All humans are mortal”, and “Socrates is human”, we wish to conclude that “Socrates is mortal.” We therefore need rules to reason about predicate expressions. Symbolically,

$$[(\forall x)(H(x) \rightarrow M(x)) \land H(s)] \rightarrow M(s)$$

Motivation

Example

From "All humans are mortal", and "Socrates is human", we wish to conclude that "Socrates is mortal." We therefore need rules to reason about predicate expressions. Symbolically,

$$[(\forall x)(H(x) \rightarrow M(x)) \wedge H(s)] \rightarrow M(s)$$

Note

All the rules of propositional logic work; however, two points need to be made:

Motivation

Example

From "All humans are mortal", and "Socrates is human", we wish to conclude that "Socrates is mortal." We therefore need rules to reason about predicate expressions. Symbolically,

$$[(\forall x)(H(x) \rightarrow M(x)) \land H(s)] \rightarrow M(s)$$

Note

All the rules of propositional logic work; however, two points need to be made:

(i) *A single atom in a predicate expression includes the quantifier.*

Motivation

Example

From "All humans are mortal", and "Socrates is human", we wish to conclude that "Socrates is mortal." We therefore need rules to reason about predicate expressions. Symbolically,

$$[(\forall x)(H(x) \rightarrow M(x)) \wedge H(s)] \rightarrow M(s)$$

Note

All the rules of propositional logic work; however, two points need to be made:

(i) *A single atom in a predicate expression includes the quantifier. For instance,*

$$[((\forall x)P(x) \rightarrow (\forall x)Q(x)) \wedge (\forall x)P(x)] \rightarrow (\forall x)Q(x)$$

is a valid argument in predicate logic.

Motivation

Example

From "All humans are mortal", and "Socrates is human", we wish to conclude that "Socrates is mortal." We therefore need rules to reason about predicate expressions. Symbolically,

$$[(\forall x)(H(x) \rightarrow M(x)) \land H(s)] \rightarrow M(s)$$

Note

All the rules of propositional logic work; however, two points need to be made:

(i) *A single atom in a predicate expression includes the quantifier. For instance,*

$$[((\forall x)P(x) \rightarrow (\forall x)Q(x)) \land (\forall x)P(x)] \rightarrow (\forall x)Q(x)$$

is a valid argument in predicate logic.

(ii) *Propositional rules are not sufficient. For instance, you cannot use propositional rules to conclude validity in the Socrates example.*

Universal Instantiation

Universal Instantiation

Details

(i) From $(\forall x)P(x)$, you can conclude $P(t)$, where t is any constant or variable.

Universal Instantiation

Details

(i) From $(\forall x)P(x)$, you can conclude $P(t)$, where t is any constant or variable.

(ii) Rule is abbreviated as u.i.

Universal Instantiation

Details

(i) From $(\forall x)P(x)$, you can conclude $P(t)$, where t is any constant or variable.

(ii) Rule is abbreviated as u.i.

(iii) If t is a variable, it must not fall within the scope of a quantifier for t.

Universal Instantiation

Details

(i) From $(\forall x)P(x)$, you can conclude $P(t)$, where t is any constant or variable.

(ii) Rule is abbreviated as u.i.

(iii) If t is a variable, it must not fall within the scope of a quantifier for t. For instance, from $(\forall x)(\exists y)P(x, y)$, you cannot conclude $(\exists y)P(y, y)$. (Domain of integers).

Universal Instantiation

Details

(i) From $(\forall x)P(x)$, you can conclude $P(t)$, where t is any constant or variable.

(ii) Rule is abbreviated as u.i.

(iii) If t is a variable, it must not fall within the scope of a quantifier for t. For instance, from $(\forall x)(\exists y)P(x,y)$, you cannot conclude $(\exists y)P(y,y)$. (Domain of integers).

Example

Let us prove that the following argument is valid, using ui.

$$[(\forall x)[H(x) \rightarrow M(x)] \land H(s)] \rightarrow M(s)$$

UI (example)

Proof.

Consider the following proof sequence:

UI (example)

Proof.

Consider the following proof sequence:

(i) $(\forall x)[H(x) \rightarrow M(x)]$ hypothesis.

UI (example)

Proof.

Consider the following proof sequence:

(i) $(\forall x)[H(x) \rightarrow M(x)]$ hypothesis.

(ii) $H(s) \rightarrow M(s)$ (i), ui.

UI (example)

Proof.

Consider the following proof sequence:

(i) $(\forall x)[H(x) \rightarrow M(x)]$ hypothesis.

(ii) $H(s) \rightarrow M(s)$ (i), ui.

(iii) $H(s)$ hypothesis.

UI (example)

Proof.

Consider the following proof sequence:

(i) $(\forall x)[H(x) \to M(x)]$ hypothesis.

(ii) $H(s) \to M(s)$ (i), ui.

(iii) $H(s)$ hypothesis.

(iv) $M(s)$ (ii), (iii), Modus Ponens.

□

UI (example)

Proof.

Consider the following proof sequence:

(i) $(\forall x)[H(x) \rightarrow M(x)]$ hypothesis.

(ii) $H(s) \rightarrow M(s)$ (i), ui.

(iii) $H(s)$ hypothesis.

(iv) $M(s)$ (ii), (iii), Modus Ponens.

□

Example

Prove that the following argument is valid.

$$[(\forall x)[P(x) \rightarrow R(x)] \wedge (R(y))'] \rightarrow (P(y))'$$

Existential Instantiation

Existential Instantiation

Details

(i) From $(\exists x)P(x)$, you can conclude $P(a)$, where a is a constant symbol not used previously in the proof sequence.

Existential Instantiation

Details

(i) From $(\exists x)P(x)$, you can conclude $P(a)$, where a is a constant symbol not used previously in the proof sequence.

(ii) Rule is abbreviated as e.i.

Existential Instantiation

Details

(i) From $(\exists x)P(x)$, you can conclude $P(a)$, where a is a constant symbol not used previously in the proof sequence.

(ii) Rule is abbreviated as e.i.

(iii) Must be the first rule that introduces a.

Existential Instantiation

Details

(i) From $(\exists x)P(x)$, you can conclude $P(a)$, where a is a constant symbol not used previously in the proof sequence.

(ii) Rule is abbreviated as e.i.

(iii) Must be the first rule that introduces a.

Example

Show that $[(\forall x)[P(x) \rightarrow Q(x)] \land (\exists y)P(y)] \rightarrow (\exists y)Q(y)$ is valid.

Existential Instantiation

Details

(i) From $(\exists x)P(x)$, you can conclude $P(a)$, where a is a constant symbol not used previously in the proof sequence.

(ii) Rule is abbreviated as e.i.

(iii) Must be the first rule that introduces a.

Example

Show that $[(\forall x)[P(x) \rightarrow Q(x)] \wedge (\exists y)P(y)] \rightarrow (\exists y)Q(y)$ is valid.
Consider the following proof sequence.

Existential Instantiation

Details

(i) From $(\exists x)P(x)$, you can conclude $P(a)$, where a is a constant symbol not used previously in the proof sequence.
(ii) Rule is abbreviated as e.i.
(iii) Must be the first rule that introduces a.

Example

Show that $[(\forall x)[P(x) \rightarrow Q(x)] \wedge (\exists y)P(y)] \rightarrow (\exists y)Q(y)$ is valid.
Consider the following proof sequence.

(i) $(\exists y)P(y)$ hypothesis.

Existential Instantiation

Details

(i) From $(\exists x)P(x)$, you can conclude $P(a)$, where a is a constant symbol not used previously in the proof sequence.
(ii) Rule is abbreviated as e.i.
(iii) Must be the first rule that introduces a.

Example

Show that $[(\forall x)[P(x) \rightarrow Q(x)] \wedge (\exists y)P(y)] \rightarrow (\exists y)Q(y)$ is valid.
Consider the following proof sequence.

(i) $(\exists y)P(y)$ hypothesis.
(ii) $P(a)$ (i), e.i.

Existential Instantiation

Details

(i) From $(\exists x)P(x)$, you can conclude $P(a)$, where a is a constant symbol not used previously in the proof sequence.
(ii) Rule is abbreviated as e.i.
(iii) Must be the first rule that introduces a.

Example

Show that $[(\forall x)[P(x) \rightarrow Q(x)] \wedge (\exists y)P(y)] \rightarrow (\exists y)Q(y)$ is valid.
Consider the following proof sequence.

(i) $(\exists y)P(y)$ hypothesis.
(ii) $P(a)$ (i), e.i.
(iii) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.

Existential Instantiation

Details

(i) From $(\exists x)P(x)$, you can conclude $P(a)$, where a is a constant symbol not used previously in the proof sequence.

(ii) Rule is abbreviated as e.i.

(iii) Must be the first rule that introduces a.

Example

Show that $[(\forall x)[P(x) \rightarrow Q(x)] \wedge (\exists y)P(y)] \rightarrow (\exists y)Q(y)$ is valid.
Consider the following proof sequence.

(i) $(\exists y)P(y)$ hypothesis.

(ii) $P(a)$ (i), e.i.

(iii) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.

(iv) $P(a) \rightarrow Q(a)$ (iii), ui.

Existential Instantiation

Details

(i) From $(\exists x)P(x)$, you can conclude $P(a)$, where a is a constant symbol not used previously in the proof sequence.

(ii) Rule is abbreviated as e.i.

(iii) Must be the first rule that introduces a.

Example

Show that $[(\forall x)[P(x) \rightarrow Q(x)] \wedge (\exists y)P(y)] \rightarrow (\exists y)Q(y)$ is valid.
Consider the following proof sequence.

(i) $(\exists y)P(y)$ hypothesis.

(ii) $P(a)$ (i), e.i.

(iii) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.

(iv) $P(a) \rightarrow Q(a)$ (iii), ui.

(v) $Q(a)$ (ii), (iv), Modus Ponens.

Existential Instantiation

Details

(i) From $(\exists x)P(x)$, you can conclude $P(a)$, where a is a constant symbol not used previously in the proof sequence.

(ii) Rule is abbreviated as e.i.

(iii) Must be the first rule that introduces a.

Example

Show that $[(\forall x)[P(x) \rightarrow Q(x)] \wedge (\exists y)P(y)] \rightarrow (\exists y)Q(y)$ is valid.
Consider the following proof sequence.

(i) $(\exists y)P(y)$ hypothesis.

(ii) $P(a)$ (i), e.i.

(iii) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.

(iv) $P(a) \rightarrow Q(a)$ (iii), ui.

(v) $Q(a)$ (ii), (iv), Modus Ponens.

Note

Steps (i)-(ii) and (iii)-(iv) **cannot** *be interchanged.*

Universal Generalization

Universal Generalization

Details

(i) From $P(x)$, you can conclude $(\forall x)P(x)$.

Universal Generalization

Details

(i) From $P(x)$, you can conclude $(\forall x)P(x)$.

(ii) Rule is abbreviated as u.g.

Universal Generalization

Details

(i) From $P(x)$, you can conclude $(\forall x)P(x)$.

(ii) Rule is abbreviated as u.g.

(iii) $P(x)$ has not been deduced from a hypothesis in which x is a free variable.

Universal Generalization

Details

(i) From $P(x)$, you can conclude $(\forall x)P(x)$.

(ii) Rule is abbreviated as u.g.

(iii) $P(x)$ has not been deduced from a hypothesis in which x is a free variable. Also, $P(x)$ has not been deduced using e.i.

Universal Generalization

Details

(i) From $P(x)$, you can conclude $(\forall x)P(x)$.

(ii) Rule is abbreviated as u.g.

(iii) $P(x)$ has not been deduced from a hypothesis in which x is a free variable. Also, $P(x)$ has not been deduced using e.i.

Example

Show that the following argument is valid.

$$[(\forall x)[P(x) \rightarrow Q(x)] \land (\forall x)P(x)] \rightarrow (\forall x)Q(x).$$

UG (example)

Proof

Consider the following proof sequence:

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.

(ii) $P(x) \rightarrow Q(x)$ (i), ui.

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.

(ii) $P(x) \rightarrow Q(x)$ (i), ui.

(iii) $(\forall x)P(x)$ hypothesis.

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.

(ii) $P(x) \rightarrow Q(x)$ (i), ui.

(iii) $(\forall x)P(x)$ hypothesis.

(iv) $P(x)$ (iii), ui.

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.

(ii) $P(x) \rightarrow Q(x)$ (i), ui.

(iii) $(\forall x)P(x)$ hypothesis.

(iv) $P(x)$ (iii), ui.

(v) $Q(x)$ (ii), (iv) Modus Ponens.

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.

(ii) $P(x) \rightarrow Q(x)$ (i), ui.

(iii) $(\forall x)P(x)$ hypothesis.

(iv) $P(x)$ (iii), ui.

(v) $Q(x)$ (ii), (iv) Modus Ponens.

(vi) $(\forall x)Q(x)$ (v), u.g. (Neither restriction has been violated.)

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.
(ii) $P(x) \rightarrow Q(x)$ (i), ui.
(iii) $(\forall x)P(x)$ hypothesis.
(iv) $P(x)$ (iii), ui.
(v) $Q(x)$ (ii), (iv) Modus Ponens.
(vi) $(\forall x)Q(x)$ (v), u.g. (Neither restriction has been violated.)

Incorrect usage of UG

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.
(ii) $P(x) \rightarrow Q(x)$ (i), ui.
(iii) $(\forall x)P(x)$ hypothesis.
(iv) $P(x)$ (iii), ui.
(v) $Q(x)$ (ii), (iv) Modus Ponens.
(vi) $(\forall x)Q(x)$ (v), u.g. (Neither restriction has been violated.)

Incorrect usage of UG

(i) $P(x)$ hypothesis.

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.
(ii) $P(x) \rightarrow Q(x)$ (i), ui.
(iii) $(\forall x)P(x)$ hypothesis.
(iv) $P(x)$ (iii), ui.
(v) $Q(x)$ (ii), (iv) Modus Ponens.
(vi) $(\forall x)Q(x)$ (v), u.g. (Neither restriction has been violated.)

Incorrect usage of UG

(i) $P(x)$ hypothesis.
(ii) $(\forall x)P(x)$ (i), u.g.

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.
(ii) $P(x) \rightarrow Q(x)$ (i), ui.
(iii) $(\forall x)P(x)$ hypothesis.
(iv) $P(x)$ (iii), ui.
(v) $Q(x)$ (ii), (iv) Modus Ponens.
(vi) $(\forall x)Q(x)$ (v), u.g. (Neither restriction has been violated.)

Incorrect usage of UG

(i) $P(x)$ hypothesis.
(ii) $(\forall x)P(x)$ (i), u.g. (Free variable rule).

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.
(ii) $P(x) \rightarrow Q(x)$ (i), ui.
(iii) $(\forall x)P(x)$ hypothesis.
(iv) $P(x)$ (iii), ui.
(v) $Q(x)$ (ii), (iv) Modus Ponens.
(vi) $(\forall x)Q(x)$ (v), u.g. (Neither restriction has been violated.)

Incorrect usage of UG

(i) $P(x)$ hypothesis.
(ii) $(\forall x)P(x)$ (i), u.g. (Free variable rule).

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.
(ii) $P(x) \rightarrow Q(x)$ (i), ui.
(iii) $(\forall x)P(x)$ hypothesis.
(iv) $P(x)$ (iii), ui.
(v) $Q(x)$ (ii), (iv) Modus Ponens.
(vi) $(\forall x)Q(x)$ (v), u.g. (Neither restriction has been violated.)

Incorrect usage of UG

(i) $P(x)$ hypothesis.
(ii) $(\forall x)P(x)$ (i), u.g. (Free variable rule).

(i) $(\forall x)(\exists y)Q(x, y)$ hypothesis.

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.
(ii) $P(x) \rightarrow Q(x)$ (i), ui.
(iii) $(\forall x)P(x)$ hypothesis.
(iv) $P(x)$ (iii), ui.
(v) $Q(x)$ (ii), (iv) Modus Ponens.
(vi) $(\forall x)Q(x)$ (v), u.g. (Neither restriction has been violated.)

Incorrect usage of UG

(i) $P(x)$ hypothesis.
(ii) $(\forall x)P(x)$ (i), u.g. (Free variable rule).

(i) $(\forall x)(\exists y)Q(x, y)$ hypothesis.
(ii) $(\exists y)Q(x, y)$ (i), ui.

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.
(ii) $P(x) \rightarrow Q(x)$ (i), ui.
(iii) $(\forall x)P(x)$ hypothesis.
(iv) $P(x)$ (iii), ui.
(v) $Q(x)$ (ii), (iv) Modus Ponens.
(vi) $(\forall x)Q(x)$ (v), u.g. (Neither restriction has been violated.)

Incorrect usage of UG

(i) $P(x)$ hypothesis.
(ii) $(\forall x)P(x)$ (i), u.g. (Free variable rule).

(i) $(\forall x)(\exists y)Q(x, y)$ hypothesis.
(ii) $(\exists y)Q(x, y)$ (i), ui.
(iii) $Q(x, a)$ (ii), e.i.

UG (example)

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \rightarrow Q(x)]$ hypothesis.

(ii) $P(x) \rightarrow Q(x)$ (i), ui.

(iii) $(\forall x)P(x)$ hypothesis.

(iv) $P(x)$ (iii), ui.

(v) $Q(x)$ (ii), (iv) Modus Ponens.

(vi) $(\forall x)Q(x)$ (v), u.g. (Neither restriction has been violated.)

Incorrect usage of UG

(i) $P(x)$ hypothesis.

(ii) $(\forall x)P(x)$ (i), u.g. (Free variable rule).

(i) $(\forall x)(\exists y)Q(x, y)$ hypothesis.

(ii) $(\exists y)Q(x, y)$ (i), ui.

(iii) $Q(x, a)$ (ii), e.i.

(iv) $(\forall x)Q(x, a)$ (iii), u.g. (Cannot use u.g., if ei is used before in sequence).

Existential Generalization

Existential Generalization

Details

(i) From $P(x)$ or $P(a)$, you can conclude $(\exists x)P(x)$.

Existential Generalization

Details

(i) From $P(x)$ or $P(a)$, you can conclude $(\exists x)P(x)$.

(ii) Rule is abbreviated as e.g.

Existential Generalization

Details

(i) From $P(x)$ or $P(a)$, you can conclude $(\exists x)P(x)$.
(ii) Rule is abbreviated as e.g.
(iii) To go from $P(a)$ to $(\exists x)P(x)$, x must not appear in $P(a)$.

Existential Generalization

Details

(i) From $P(x)$ or $P(a)$, you can conclude $(\exists x)P(x)$.

(ii) Rule is abbreviated as e.g.

(iii) To go from $P(a)$ to $(\exists x)P(x)$, x must not appear in $P(a)$. Otherwise, we could derive $(\exists y)Q(y, y)$ from $Q(a, y)$!

Existential Generalization

Details

(i) From $P(x)$ or $P(a)$, you can conclude $(\exists x)P(x)$.

(ii) Rule is abbreviated as e.g.

(iii) To go from $P(a)$ to $(\exists x)P(x)$, x must not appear in $P(a)$. Otherwise, we could derive $(\exists y)Q(y,y)$ from $Q(a,y)$! The argument $Q(a,y) \rightarrow (\exists y)Q(y,y)$ is simply not valid. (Why?)

Existential Generalization

Details

(i) From $P(x)$ or $P(a)$, you can conclude $(\exists x)P(x)$.

(ii) Rule is abbreviated as e.g.

(iii) To go from $P(a)$ to $(\exists x)P(x)$, x must not appear in $P(a)$. Otherwise, we could derive $(\exists y)Q(y, y)$ from $Q(a, y)$! The argument $Q(a, y) \rightarrow (\exists y)Q(y, y)$ is simply not valid. (Why?)

Main points of predicate rules

Existential Generalization

Details

(i) From $P(x)$ or $P(a)$, you can conclude $(\exists x)P(x)$.

(ii) Rule is abbreviated as e.g.

(iii) To go from $P(a)$ to $(\exists x)P(x)$, x must not appear in $P(a)$. Otherwise, we could derive $(\exists y)Q(y, y)$ from $Q(a, y)$! The argument $Q(a, y) \to (\exists y)Q(y, y)$ is simply not valid. (Why?)

Main points of predicate rules

(i) Strip off quantifiers.

Existential Generalization

Details

(i) From $P(x)$ or $P(a)$, you can conclude $(\exists x)P(x)$.

(ii) Rule is abbreviated as e.g.

(iii) To go from $P(a)$ to $(\exists x)P(x)$, x must not appear in $P(a)$. Otherwise, we could derive $(\exists y)Q(y, y)$ from $Q(a, y)$! The argument $Q(a, y) \rightarrow (\exists y)Q(y, y)$ is simply not valid. (Why?)

Main points of predicate rules

(i) Strip off quantifiers.

(ii) Work with separate wffs.

Existential Generalization

Details

(i) From $P(x)$ or $P(a)$, you can conclude $(\exists x)P(x)$.

(ii) Rule is abbreviated as e.g.

(iii) To go from $P(a)$ to $(\exists x)P(x)$, x must not appear in $P(a)$. Otherwise, we could derive $(\exists y)Q(y,y)$ from $Q(a,y)$! The argument $Q(a,y) \rightarrow (\exists y)Q(y,y)$ is simply not valid. (Why?)

Main points of predicate rules

(i) Strip off quantifiers.

(ii) Work with separate wffs.

(iii) Insert quantifiers as necessary.

Some more examples

Example

Show that the following arguments are valid:

1. $(\forall x)[P(x) \land Q(x)] \rightarrow (\forall x)P(x) \land (\forall x)Q(x)$.
2. $[(\forall y)[P(x) \rightarrow Q(x,y)]] \rightarrow [P(x) \rightarrow (\forall y)Q(x,y)]$.
3. $[P(x) \rightarrow (\forall y)Q(x,y)] \rightarrow (\forall y)[P(x) \rightarrow Q(x,y)]$.

Proof of Example 1

Proof

Consider the following proof sequence:

Proof of Example 1

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \wedge Q(x)]$ hypothesis.

Proof of Example 1

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \wedge Q(x)]$ hypothesis.

(ii) $P(x) \wedge Q(x)$ (i), ui.

Proof of Example 1

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \land Q(x)]$ hypothesis.

(ii) $P(x) \land Q(x)$ (i), ui.

(iii) $P(x)$ (ii), Simplification.

Proof of Example 1

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \wedge Q(x)]$ hypothesis.

(ii) $P(x) \wedge Q(x)$ (i), ui.

(iii) $P(x)$ (ii), Simplification.

(iv) $(\forall x)P(x)$ (iii), u.g.

Proof of Example 1

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \wedge Q(x)]$ hypothesis.
(ii) $P(x) \wedge Q(x)$ (i), ui.
(iii) $P(x)$ (ii), Simplification.
(iv) $(\forall x)P(x)$ (iii), u.g.
(v) $Q(x)$ (ii), Simplification.

Proof of Example 1

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \wedge Q(x)]$ hypothesis.

(ii) $P(x) \wedge Q(x)$ (i), ui.

(iii) $P(x)$ (ii), Simplification.

(iv) $(\forall x)P(x)$ (iii), u.g.

(v) $Q(x)$ (ii), Simplification.

(vi) $(\forall x)Q(x)$ (v), u.g.

Proof of Example 1

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \land Q(x)]$ hypothesis.

(ii) $P(x) \land Q(x)$ (i), ui.

(iii) $P(x)$ (ii), Simplification.

(iv) $(\forall x)P(x)$ (iii), u.g.

(v) $Q(x)$ (ii), Simplification.

(vi) $(\forall x)Q(x)$ (v), u.g.

(vii) $(\forall x)P(x) \land (\forall x)Q(x)$ (iv), (vi), Conjunction.

Proof of Example 1

Proof

Consider the following proof sequence:

(i) $(\forall x)[P(x) \land Q(x)]$ hypothesis.
(ii) $P(x) \land Q(x)$ (i), ui.
(iii) $P(x)$ (ii), Simplification.
(iv) $(\forall x)P(x)$ (iii), u.g.
(v) $Q(x)$ (ii), Simplification.
(vi) $(\forall x)Q(x)$ (v), u.g.
(vii) $(\forall x)P(x) \land (\forall x)Q(x)$ (iv), (vi), Conjunction.

Note

Note that neither restriction has been violated in the u.g. steps.

Proof of Example 2

Proof

Using the Deduction Method, rewrite the argument as:

$$[(\forall y)[P(x) \rightarrow Q(x,y)] \wedge P(x)] \rightarrow (\forall y)Q(x,y)$$

Consider the following proof sequence:

Proof of Example 2

Proof

Using the Deduction Method, rewrite the argument as:

$$[(\forall y)[P(x) \rightarrow Q(x,y)] \land P(x)] \rightarrow (\forall y)Q(x,y)$$

Consider the following proof sequence:

(i) $(\forall y)[P(x) \rightarrow Q(x,y)]$ hypothesis.

Proof of Example 2

Proof

Using the Deduction Method, rewrite the argument as:

$$[(\forall y)[P(x) \rightarrow Q(x,y)] \land P(x)] \rightarrow (\forall y)Q(x,y)$$

Consider the following proof sequence:

(i) $(\forall y)[P(x) \rightarrow Q(x,y)]$ hypothesis.

(ii) $P(x) \rightarrow Q(x,y)$ (i), ui.

Proof of Example 2

Proof

Using the Deduction Method, rewrite the argument as:

$$[(\forall y)[P(x) \rightarrow Q(x,y)] \land P(x)] \rightarrow (\forall y)Q(x,y)$$

Consider the following proof sequence:

(i) $(\forall y)[P(x) \rightarrow Q(x,y)]$ hypothesis.

(ii) $P(x) \rightarrow Q(x,y)$ (i), ui.

(iii) $P(x)$ hypothesis.

Proof of Example 2

Proof

Using the Deduction Method, rewrite the argument as:

$$[(\forall y)[P(x) \rightarrow Q(x,y)] \land P(x)] \rightarrow (\forall y)Q(x,y)$$

Consider the following proof sequence:

(i) $(\forall y)[P(x) \rightarrow Q(x,y)]$ hypothesis.
(ii) $P(x) \rightarrow Q(x,y)$ (i), ui.
(iii) $P(x)$ hypothesis.
(iv) $Q(x,y)$ (ii), (iii), Modus Ponens.

Proof of Example 2

Proof

Using the Deduction Method, rewrite the argument as:

$$[(\forall y)[P(x) \rightarrow Q(x,y)] \land P(x)] \rightarrow (\forall y)Q(x,y)$$

Consider the following proof sequence:

(i) $(\forall y)[P(x) \rightarrow Q(x,y)]$ hypothesis.

(ii) $P(x) \rightarrow Q(x,y)$ (i), ui.

(iii) $P(x)$ hypothesis.

(iv) $Q(x,y)$ (ii), (iii), Modus Ponens.

(v) $(\forall y)Q(x,y)$ (iv), u.g.

Proof of Example 3

Proof

Proof of Example 3

Proof

We will need a new technique called “temporary hypothesis”. Consider the following proof sequence:

Proof of Example 3

Proof

We will need a new technique called "temporary hypothesis". Consider the following proof sequence:

(i) $P(x) \rightarrow (\forall y)Q(x, y)$ hypothesis.

Proof of Example 3

Proof

We will need a new technique called “temporary hypothesis”. Consider the following proof sequence:

(i) $P(x) \rightarrow (\forall y)Q(x, y)$ hypothesis.

(ii) $P(x)$ temporary hypothesis.

Proof of Example 3

Proof

We will need a new technique called "temporary hypothesis". Consider the following proof sequence:

(i) $P(x) \rightarrow (\forall y)Q(x,y)$ hypothesis.

(ii) $P(x)$ temporary hypothesis.

(iii) $(\forall y)Q(x,y)$ (i), (ii), Modus Ponens.

Proof of Example 3

Proof

We will need a new technique called "temporary hypothesis". Consider the following proof sequence:

(i) $P(x) \rightarrow (\forall y)Q(x,y)$ hypothesis.

(ii) $P(x)$ temporary hypothesis.

(iii) $(\forall y)Q(x,y)$ (i), (ii), Modus Ponens.

(iv) $Q(x,y)$ (iii), ui.

Proof of Example 3

Proof

We will need a new technique called "temporary hypothesis". Consider the following proof sequence:

(i) $P(x) \rightarrow (\forall y)Q(x,y)$ hypothesis.

(ii) $P(x)$ temporary hypothesis.

(iii) $(\forall y)Q(x,y)$ (i), (ii), Modus Ponens.

(iv) $Q(x,y)$ (iii), ui.

(v) $P(x) \rightarrow Q(x,y)$ temporary hypothesis discharged.

Proof of Example 3

Proof

We will need a new technique called "temporary hypothesis". Consider the following proof sequence:

(i) $P(x) \rightarrow (\forall y)Q(x,y)$ hypothesis.

(ii) $P(x)$ temporary hypothesis.

(iii) $(\forall y)Q(x,y)$ (i), (ii), Modus Ponens.

(iv) $Q(x,y)$ (iii), ui.

(v) $P(x) \rightarrow Q(x,y)$ temporary hypothesis discharged.

(vi) $(\forall y)[P(x) \rightarrow Q(x,y)]$ (v), u.g.

Two laws

Negation

Two laws

Negation

(i) $[(\exists x)A(x)]' \Leftrightarrow (\forall x)[A(x)]'$.

Two laws

Negation

(i) $[(\exists x)A(x)]' \Leftrightarrow (\forall x)[A(x)]'$.

(ii) $[(\forall x)A(x)]' \Leftrightarrow (\exists x)[A(x)]'$.

Two laws

Negation

(i) $[(\exists x)A(x)]' \Leftrightarrow (\forall x)[A(x)]'$.

(ii) $[(\forall x)A(x)]' \Leftrightarrow (\exists x)[A(x)]'$.

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port.

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port. Symbolically,

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port. Symbolically,

$$[(\forall x)[M(x) \rightarrow S(x)] \wedge (\exists x)[M(x) \wedge P(x)]] \rightarrow (\exists x)[M(x) \wedge S(x) \wedge P(x)].$$

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port. Symbolically,

$$[(\forall x)[M(x) \rightarrow S(x)] \land (\exists x)[M(x) \land P(x)]] \rightarrow (\exists x)[M(x) \land S(x) \land P(x)].$$

Proof.

Consider the following proof sequence:

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port. Symbolically,

$$[(\forall x)[M(x) \rightarrow S(x)] \wedge (\exists x)[M(x) \wedge P(x)]] \rightarrow (\exists x)[M(x) \wedge S(x) \wedge P(x)].$$

Proof.

Consider the following proof sequence:

(i) $(\forall x)[M(x) \rightarrow S(x)]$ hypothesis.

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port.
Symbolically,

$$[(\forall x)[M(x) \rightarrow S(x)] \land (\exists x)[M(x) \land P(x)]] \rightarrow (\exists x)[M(x) \land S(x) \land P(x)].$$

Proof.

Consider the following proof sequence:

(i) $(\forall x)[M(x) \rightarrow S(x)]$ hypothesis.

(ii) $(\exists x)[M(x) \land P(x)]$ hypothesis.

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port. Symbolically,

$$[(\forall x)[M(x) \rightarrow S(x)] \land (\exists x)[M(x) \land P(x)]] \rightarrow (\exists x)[M(x) \land S(x) \land P(x)].$$

Proof.

Consider the following proof sequence:

(i) $(\forall x)[M(x) \rightarrow S(x)]$ hypothesis.

(ii) $(\exists x)[M(x) \land P(x)]$ hypothesis.

(iii) $M(a) \land P(a)$ (ii), ei.

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port.
Symbolically,

$$[(\forall x)[M(x) \rightarrow S(x)] \wedge (\exists x)[M(x) \wedge P(x)]] \rightarrow (\exists x)[M(x) \wedge S(x) \wedge P(x)].$$

Proof.

Consider the following proof sequence:

(i) $(\forall x)[M(x) \rightarrow S(x)]$ hypothesis.

(ii) $(\exists x)[M(x) \wedge P(x)]$ hypothesis.

(iii) $M(a) \wedge P(a)$ (ii), ei.

(iv) $M(a) \rightarrow S(a)$ (i), ui.

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port. Symbolically,

$$[(\forall x)[M(x) \rightarrow S(x)] \wedge (\exists x)[M(x) \wedge P(x)]] \rightarrow (\exists x)[M(x) \wedge S(x) \wedge P(x)].$$

Proof.

Consider the following proof sequence:

(i) $(\forall x)[M(x) \rightarrow S(x)]$ hypothesis.
(ii) $(\exists x)[M(x) \wedge P(x)]$ hypothesis.
(iii) $M(a) \wedge P(a)$ (ii), ei.
(iv) $M(a) \rightarrow S(a)$ (i), ui.
(v) $M(a)$ (iii), Simplification.

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port. Symbolically,

$$[(\forall x)[M(x) \rightarrow S(x)] \wedge (\exists x)[M(x) \wedge P(x)]] \rightarrow (\exists x)[M(x) \wedge S(x) \wedge P(x)].$$

Proof.

Consider the following proof sequence:

(i) $(\forall x)[M(x) \rightarrow S(x)]$ hypothesis.

(ii) $(\exists x)[M(x) \wedge P(x)]$ hypothesis.

(iii) $M(a) \wedge P(a)$ (ii), ei.

(iv) $M(a) \rightarrow S(a)$ (i), ui.

(v) $M(a)$ (iii), Simplification.

(vi) $S(a)$ (iv), (v), Modus Ponens.

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port. Symbolically,

$$[(\forall x)[M(x) \rightarrow S(x)] \land (\exists x)[M(x) \land P(x)]] \rightarrow (\exists x)[M(x) \land S(x) \land P(x)].$$

Proof.

Consider the following proof sequence:

(i) $(\forall x)[M(x) \rightarrow S(x)]$ hypothesis.

(ii) $(\exists x)[M(x) \land P(x)]$ hypothesis.

(iii) $M(a) \land P(a)$ (ii), ei.

(iv) $M(a) \rightarrow S(a)$ (i), ui.

(v) $M(a)$ (iii), Simplification.

(vi) $S(a)$ (iv), (v), Modus Ponens.

(vii) $M(a) \land P(a) \land S(a)$ (iii), (vi), Conjunction.

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port.
Symbolically,

$$[(\forall x)[M(x) \rightarrow S(x)] \wedge (\exists x)[M(x) \wedge P(x)]] \rightarrow (\exists x)[M(x) \wedge S(x) \wedge P(x)].$$

Proof.

Consider the following proof sequence:

(i) $(\forall x)[M(x) \rightarrow S(x)]$ hypothesis.
(ii) $(\exists x)[M(x) \wedge P(x)]$ hypothesis.
(iii) $M(a) \wedge P(a)$ (ii), ei.
(iv) $M(a) \rightarrow S(a)$ (i), ui.
(v) $M(a)$ (iii), Simplification.
(vi) $S(a)$ (iv), (v), Modus Ponens.
(vii) $M(a) \wedge P(a) \wedge S(a)$ (iii), (vi), Conjunction.
(viii) $M(a) \wedge S(a) \wedge P(a)$ (vii), commutativity.

A verbal argument

Example

Every microcomputer has a serial interface port. Some microcomputers have a parallel port. Therefore, some microcomputers have both a serial and a parallel port.
Symbolically,

$$[(\forall x)[M(x) \rightarrow S(x)] \wedge (\exists x)[M(x) \wedge P(x)]] \rightarrow (\exists x)[M(x) \wedge S(x) \wedge P(x)].$$

Proof.

Consider the following proof sequence:

(i) $(\forall x)[M(x) \rightarrow S(x)]$ hypothesis.
(ii) $(\exists x)[M(x) \wedge P(x)]$ hypothesis.
(iii) $M(a) \wedge P(a)$ (ii), ei.
(iv) $M(a) \rightarrow S(a)$ (i), ui.
(v) $M(a)$ (iii), Simplification.
(vi) $S(a)$ (iv), (v), Modus Ponens.
(vii) $M(a) \wedge P(a) \wedge S(a)$ (iii), (vi), Conjunction.
(viii) $M(a) \wedge S(a) \wedge P(a)$ (vii), commutativity.
(ix) $(\exists x)[M(x) \wedge S(x) \wedge P(x)]$ (viii), e.g.

□

The Semantic Argument Method

The Semantic Argument Method

Main Idea

The Semantic Argument Method

Main Idea

The principal proof technique is contradiction.

The Semantic Argument Method

Main Idea

The principal proof technique is contradiction. In order to prove the validity of a formula F, we assume the existence of an interpretation I, such that $I \not\models F$.

The Semantic Argument Method

Main Idea

The principal proof technique is contradiction. In order to prove the validity of a formula F, we assume the existence of an interpretation I, such that $I \not\models F$. Under this assumption, we construct the interpretation tree, by applying **proof rules**.

The Semantic Argument Method

Main Idea

The principal proof technique is contradiction. In order to prove the validity of a formula F, we assume the existence of an interpretation I, such that $I \not\models F$. Under this assumption, we construct the interpretation tree, by applying **proof rules**. As inferences are drawn, the branches of the tree may close, because a contradiction ($\perp$) is derived.

The Semantic Argument Method

Main Idea

The principal proof technique is contradiction. In order to prove the validity of a formula F, we assume the existence of an interpretation I, such that $I \not\models F$. Under this assumption, we construct the interpretation tree, by applying **proof rules**. As inferences are drawn, the branches of the tree may close, because a contradiction ($\perp$) is derived. If every branch is closed, F must be valid.

The Semantic Argument Method

Main Idea

The principal proof technique is contradiction. In order to prove the validity of a formula F, we assume the existence of an interpretation I, such that $I \not\models F$. Under this assumption, we construct the interpretation tree, by applying **proof rules**. As inferences are drawn, the branches of the tree may close, because a contradiction ($\perp$) is derived. If every branch is closed, F must be valid. If there exists even one open branch, F is not valid.

Proof Rules

Proof Rules

Rules

Proof Rules

Rules

Proof Rules

Rules

(i) From $I \models (\forall x)\ F$, you can deduce, $I \triangleleft \{x \mapsto v\} \models F$, for some $v \in D_I$.

Proof Rules

Rules

(i) From $I \models (\forall x)\ F$, you can deduce, $I \triangleleft \{x \mapsto v\} \models F$, for some $v \in D_I$.
(ii) From $I \not\models (\exists x)\ F$, you can deduce, $I \triangleleft \{x \mapsto v\} \not\models F$, for any $v \in D_I$.

Proof Rules

Rules

(i) From $I \models (\forall x)\ F$, you can deduce, $I \lhd \{x \mapsto v\} \models F$, for some $v \in D_I$.

(ii) From $I \not\models (\exists x)\ F$, you can deduce, $I \lhd \{x \mapsto v\} \not\models F$, for any $v \in D_I$.

(iii) From $I \models (\exists x)\ F$, you can deduce, $I \lhd \{x \mapsto v\} \models F$, for a fresh $v \in D_I$.

Proof Rules

Rules

(i) From $I \models (\forall x)\ F$, you can deduce, $I \lhd \{x \mapsto v\} \models F$, for some $v \in D_I$.
(ii) From $I \not\models (\exists x)\ F$, you can deduce, $I \lhd \{x \mapsto v\} \not\models F$, for any $v \in D_I$.
(iii) From $I \models (\exists x)\ F$, you can deduce, $I \lhd \{x \mapsto v\} \models F$, for a fresh $v \in D_I$.
(iv) From $I \not\models (\forall x)\ F$, you can deduce, $I \lhd \{x \mapsto v\} \not\models F$, for a fresh $v \in D_I$.

Proof Rules

Rules

(i) From $I \models (\forall x)\ F$, you can deduce, $I \lhd \{x \mapsto v\} \models F$, for some $v \in D_I$.

(ii) From $I \not\models (\exists x)\ F$, you can deduce, $I \lhd \{x \mapsto v\} \not\models F$, for any $v \in D_I$.

(iii) From $I \models (\exists x)\ F$, you can deduce, $I \lhd \{x \mapsto v\} \models F$, for a fresh $v \in D_I$.

(iv) From $I \not\models (\forall x)\ F$, you can deduce, $I \lhd \{x \mapsto v\} \not\models F$, for a fresh $v \in D_I$.

(v) Contradiction -

Proof Rules

Rules

(i) From $I \models (\forall x)\ F$, you can deduce, $I \triangleleft \{x \mapsto v\} \models F$, for some $v \in D_I$.

(ii) From $I \not\models (\exists x)\ F$, you can deduce, $I \triangleleft \{x \mapsto v\} \not\models F$, for any $v \in D_I$.

(iii) From $I \models (\exists x)\ F$, you can deduce, $I \triangleleft \{x \mapsto v\} \models F$, for a fresh $v \in D_I$.

(iv) From $I \not\models (\forall x)\ F$, you can deduce, $I \triangleleft \{x \mapsto v\} \not\models F$, for a fresh $v \in D_I$.

(v) Contradiction - A contradiction is obtained when two variants of the original interpretation I disagree on the truth value of an n-ary predicate p, for a given tuple of domain values.

Applying the rules

Applying the rules

Example

Applying the rules

Example

Applying the rules

Example

(i) $(\forall x)\ P(x) \rightarrow (\forall y)\ P(y)$.

Applying the rules

Example

(i) $(\forall x)\ P(x) \rightarrow (\forall y)\ P(y)$.

(ii) $(\forall x)\ P(x) \rightarrow \neg(\exists x)\ \neg P(x)$.

Applying the rules

Example

(i) $(\forall x)\ P(x) \rightarrow (\forall y)\ P(y)$.

(ii) $(\forall x)\ P(x) \rightarrow \neg(\exists x)\ \neg P(x)$.

(iii) $P(a) \rightarrow (\exists x)\ P(x)$.