First Order Theories - Basic Concepts

K. Subramani[1]

[1]Lane Department of Computer Science and Electrical Engineering
West Virginia University

22 February 2013

Outline

Outline

Motivation

Motivation

Why study theories?

Motivation

Why study theories?

(i) To reason about software in particular application domains,

Motivation

Why study theories?

(i) To reason about software in particular application domains, e.g., numbers, lists, arrays.

Motivation

Why study theories?

(i) To reason about software in particular application domains, e.g., numbers, lists, arrays.

(ii) First-order theories formalize the above structures to enable reasoning.

Motivation

Why study theories?

(i) To reason about software in particular application domains, e.g., numbers, lists, arrays.
(ii) First-order theories formalize the above structures to enable reasoning.
(iii) Fragments of theories may be efficiently decidable.

Main concepts

Main concepts

Definition

A first order theory T is defined by the following components:

Main concepts

Definition

A first order theory T is defined by the following components:

(i) Its signature Σ, which is a set of constant, function and predicate symbols.

Main concepts

Definition

A first order theory T is defined by the following components:

(i) Its signature Σ, which is a set of constant, function and predicate symbols.

(ii) Its set of axioms $\mathcal{A}$, which is a set of closed FOL formulae in which only the constant, function and predicate symbols of Σ appear.

Main concepts

Definition

A first order theory T is defined by the following components:

(i) Its signature Σ, which is a set of constant, function and predicate symbols.

(ii) Its set of axioms $\mathcal{A}$, which is a set of closed FOL formulae in which only the constant, function and predicate symbols of Σ appear.

Note

A Σ-formula is constructed from constant, function and predicate symbols of Σ, as well as variables, logical connectives and quantifiers.

Main concepts

Definition

A first order theory T is defined by the following components:

(i) Its signature Σ, which is a set of constant, function and predicate symbols.

(ii) Its set of axioms $\mathcal{A}$, which is a set of closed FOL formulae in which only the constant, function and predicate symbols of Σ appear.

Note

A Σ-formula is constructed from constant, function and predicate symbols of Σ, as well as variables, logical connectives and quantifiers. The formulas themselves are syntactic identities bereft of meaning.

Main concepts

Definition

A first order theory T is defined by the following components:

(i) Its signature Σ, which is a set of constant, function and predicate symbols.

(ii) Its set of axioms $\mathcal{A}$, which is a set of closed FOL formulae in which only the constant, function and predicate symbols of Σ appear.

Note

A Σ-formula is constructed from constant, function and predicate symbols of Σ, as well as variables, logical connectives and quantifiers. The formulas themselves are syntactic identities bereft of meaning. Meaning is provided by the axiom set $\mathcal{A}$.

Fundamentals

Fundamentals

Definition

A Σ-formula F is valid in the theory T, or T-valid,

Fundamentals

Definition

A Σ-formula F is valid in the theory T, or T-valid, if every interpretation I that satisfies the axioms of T, also satisfies F.

Fundamentals

Definition

A Σ-formula F is valid in the theory T, or T-valid, if every interpretation I that satisfies the axioms of T, also satisfies F.
In other words,

$$\forall I,\ I \models \mathcal{A} \text{ implies } I \models F.$$

Fundamentals

Definition

A Σ-formula F is valid in the theory T, or T-valid, if every interpretation I that satisfies the axioms of T, also satisfies F.
In other words,

$$\forall I,\ I \models \mathcal{A} \text{ implies } I \models F.$$

We also write this as $T \models F$.

Fundamentals

Definition

A Σ-formula F is valid in the theory T, or T-valid, if every interpretation I that satisfies the axioms of T, also satisfies F.
In other words,

$$\forall I,\ I \models \mathcal{A} \text{ implies } I \models F.$$

We also write this as $T \models F$. Note that $I \models \mathcal{A}$ is short for $I \models A,\ \forall A \in \mathcal{A}$.

Fundamentals

Definition

A Σ-formula F is valid in the theory T, or T-valid, if every interpretation I that satisfies the axioms of T, also satisfies F.
In other words,

$$\forall I,\ I \models \mathcal{A} \text{ implies } I \models F.$$

We also write this as $T \models F$. Note that $I \models \mathcal{A}$ is short for $I \models A,\ \forall A \in \mathcal{A}$. An interpretation which satisfies all the axioms of $\mathcal{A}$ is called a T-interpretation.

Fundamentals

Definition

A Σ-formula F is valid in the theory T, or T-valid, if every interpretation I that satisfies the axioms of T, also satisfies F.
In other words,

$$\forall I,\ I \models \mathcal{A} \text{ implies } I \models F.$$

We also write this as $T \models F$. Note that $I \models \mathcal{A}$ is short for $I \models A,\ \forall A \in \mathcal{A}$. An interpretation which satisfies all the axioms of $\mathcal{A}$ is called a T-interpretation.

Definition

A Σ-formula F is satisfiable in the theory T, or T-satisfiable,

Fundamentals

Definition

A Σ-formula F is valid in the theory T, or T-valid, if every interpretation I that satisfies the axioms of T, also satisfies F.
In other words,

$$\forall I,\ I \models \mathcal{A} \text{ implies } I \models F.$$

We also write this as $T \models F$. Note that $I \models \mathcal{A}$ is short for $I \models A,\ \forall A \in \mathcal{A}$. An interpretation which satisfies all the axioms of $\mathcal{A}$ is called a T-interpretation.

Definition

A Σ-formula F is satisfiable in the theory T, or T-satisfiable, if there is some T-interpretation I that satisfies F.

Completeness and Consistency

Completeness and Consistency

Definition

A theory T is complete, if for every closed Σ-formula F, either $T \models F$ or $T \models \neg F$.

Completeness and Consistency

Definition

A theory T is complete, if for every closed Σ-formula F, either $T \models F$ or $T \models \neg F$.

Note

There are theories which are not complete.

Completeness and Consistency

Definition

A theory T is complete, if for every closed Σ-formula F, either $T \models F$ or $T \models \neg F$.

Note

There are theories which are not complete. Name one.

Completeness and Consistency

Definition

A theory T is complete, if for every closed Σ-formula F, either $T \models F$ or $T \models \neg F$.

Note

There are theories which are not complete. Name one.

Definition

A theory T is consistent, if there is at least one T-interpretation.

Completeness and Consistency

Definition

A theory T is complete, if for every closed Σ-formula F, either $T \models F$ or $T \models \neg F$.

Note

There are theories which are not complete. Name one.

Definition

A theory T is consistent, if there is at least one T-interpretation.

Note

An alternative definition of consistency is that there does not exist a Σ-formula F, such that $T \models F$ and $T \models \not F$.

Completeness and Consistency

Definition

A theory T is complete, if for every closed Σ-formula F, either $T \models F$ or $T \models \neg F$.

Note

There are theories which are not complete. Name one.

Definition

A theory T is consistent, if there is at least one T-interpretation.

Note

An alternative definition of consistency is that there does not exist a Σ-formula F, such that $T \models F$ and $T \models \not F$.

Definition

Two formulae F_1 and F_2 are equivalent in theory T, or T-equivalent,

Completeness and Consistency

Definition

A theory T is complete, if for every closed Σ-formula F, either $T \models F$ or $T \models \neg F$.

Note

There are theories which are not complete. Name one.

Definition

A theory T is consistent, if there is at least one T-interpretation.

Note

An alternative definition of consistency is that there does not exist a Σ-formula F, such that $T \models F$ and $T \models \not F$.

Definition

Two formulae F_1 and F_2 are equivalent in theory T, or T-equivalent, if $T \models F_1 \leftrightarrow F_2$.

Completeness and Consistency

Definition

A theory T is complete, if for every closed Σ-formula F, either $T \models F$ or $T \models \neg F$.

Note

There are theories which are not complete. Name one.

Definition

A theory T is consistent, if there is at least one T-interpretation.

Note

An alternative definition of consistency is that there does not exist a Σ-formula F, such that $T \models F$ and $T \models \not F$.

Definition

Two formulae F_1 and F_2 are equivalent in theory T, or T-equivalent, if $T \models F_1 \leftrightarrow F_2$. In other words, for every T-interpretation I, we must have, $I \models F_1$ if and only if $I \models F_2$.

Fragments

Fragments

Definition

A fragment of a theory is a syntactically-restricted subset of formulae of the theory.

Fragments

Definition

A fragment of a theory is a syntactically-restricted subset of formulae of the theory.

Example

The quantifier-free fragment of a theory T is the set of formulae without quantifiers that are valid in T.

Fragments

Definition

A fragment of a theory is a syntactically-restricted subset of formulae of the theory.

Example

The quantifier-free fragment of a theory T is the set of formulae without quantifiers that are valid in T.

Definition

A theory T is decidable if $T \models F$ is decidable for every Σ-formula F.

Combinations of theories

Combinations of theories

Definition

The **union** of two theories T_1 and T_2, denoted by $T_1 \cup T_2$ has signature $\Sigma_1 \cup \Sigma_2$ and axioms $\mathcal{A}_1 \cup \mathcal{A}_2$.

Combinations of theories

Definition

The **union** of two theories T_1 and T_2, denoted by $T_1 \cup T_2$ has signature $\Sigma_1 \cup \Sigma_2$ and axioms $\mathcal{A}_1 \cup \mathcal{A}_2$.

Note

A $(T_1 \cup T_2)$-interpretation is both a T_1-interpretation and a T_2-interpretation.

Combinations of theories

Definition

The **union** of two theories T_1 and T_2, denoted by $T_1 \cup T_2$ has signature $\Sigma_1 \cup \Sigma_2$ and axioms $\mathcal{A}_1 \cup \mathcal{A}_2$.

Note

A $(T_1 \cup T_2)$-interpretation is both a T_1-interpretation and a T_2-interpretation. A formula that is T_1-valid or T_2-valid is $(T_1 \cup T_2)$-valid.

Combinations of theories

Definition

The **union** of two theories T_1 and T_2, denoted by $T_1 \cup T_2$ has signature $\Sigma_1 \cup \Sigma_2$ and axioms $\mathcal{A}_1 \cup \mathcal{A}_2$.

Note

A $(T_1 \cup T_2)$-interpretation is both a T_1-interpretation and a T_2-interpretation. A formula that is T_1-valid or T_2-valid is $(T_1 \cup T_2)$-valid. A formula that is $(T_1 \cup T_2)$-satisfiable is both T_1-satisfiable and T_2-satisfiable.

Combinations of theories

Definition

The **union** of two theories T_1 and T_2, denoted by $T_1 \cup T_2$ has signature $\Sigma_1 \cup \Sigma_2$ and axioms $\mathcal{A}_1 \cup \mathcal{A}_2$.

Note

A $(T_1 \cup T_2)$-interpretation is both a T_1-interpretation and a T_2-interpretation. A formula that is T_1-valid or T_2-valid is $(T_1 \cup T_2)$-valid. A formula that is $(T_1 \cup T_2)$-satisfiable is both T_1-satisfiable and T_2-satisfiable.

Observation

FOL is the empty theory, i.e., the theory with no axioms.