First Order Theories - Natural numbers and Integers

K. Subramani[1]

[1]Lane Department of Computer Science and Electrical Engineering
West Virginia University

22 February, March 1, March 4 2013

Outline

Outline

Outline

Outline

Introducton

Introducton

Motivating points

Arithmetic involving addition and multiplication over the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots, \}$ has been studied for centuries.

Introducton

Motivating points

Arithmetic involving addition and multiplication over the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots, \}$ has been studied for centuries.

We focus on the following theories:

Introducton

Motivating points

Arithmetic involving addition and multiplication over the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots, \}$ has been studied for centuries.

We focus on the following theories:

(i) **Peano arithmetic** that permits addition and multiplication over natural numbers.

Introducton

Motivating points

Arithmetic involving addition and multiplication over the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots, \}$ has been studied for centuries.

We focus on the following theories:

(i) **Peano arithmetic** that permits addition and multiplication over natural numbers.

(ii) **Presburger arithmetic** that permits addition but not multiplication over the natural numbers.

Introducton

Motivating points

Arithmetic involving addition and multiplication over the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots, \}$ has been studied for centuries.

We focus on the following theories:

(i) **Peano arithmetic** that permits addition and multiplication over natural numbers.

(ii) **Presburger arithmetic** that permits addition but not multiplication over the natural numbers.

(iii) **Theory of integers** that permits over addition over the set $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \}$.

Peano Arithmetic (PA)

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \ : \ \{0, 1, +, \cdot, =\}$$

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \ : \ \{0, 1, +, \cdot, =\}$$

where,

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \ : \ \{0, 1, +, \cdot, =\}$$

where,

(i) 0 and 1 are constants.

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \ : \ \{0, 1, +, \cdot, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ and $\cdot$ are binary functions.

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \ : \ \{0, 1, +, \cdot, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ and $\cdot$ are binary functions.

(iii) $=$ is a binary predicate.

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \ : \ \{0, 1, +, \cdot, =\}$$

where,

(i) 0 and 1 are constants.
(ii) $+$ and $\cdot$ are binary functions.
(iii) $=$ is a binary predicate.

Its axiom set is the following:

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \ : \ \{0, 1, +, \cdot, =\}$$

where,

(i) 0 and 1 are constants.
(ii) $+$ and $\cdot$ are binary functions.
(iii) $=$ is a binary predicate.

Its axiom set is the following:

($\mathcal{A}$1.) $(\forall x)\ \neg(x + 1) = 0$.

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \;:\; \{0, 1, +, \cdot, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ and $\cdot$ are binary functions.

(iii) $=$ is a binary predicate.

Its axiom set is the following:

($\mathcal{A}1.$) $(\forall x)\ \neg(x+1) = 0.$

($\mathcal{A}2.$) $(\forall x)(\forall y)\ [(x+1) = (y+1)] \rightarrow (x = y).$

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \; : \; \{0, 1, +, \cdot, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ and $\cdot$ are binary functions.

(iii) $=$ is a binary predicate.

Its axiom set is the following:

($\mathcal{A}1.$) $(\forall x)\ \neg(x + 1) = 0$.

($\mathcal{A}2.$) $(\forall x)(\forall y)\ [(x + 1) = (y + 1)] \rightarrow (x = y)$.

($\mathcal{A}3.$) $(F[0] \wedge (\forall x)\ (F[x] \rightarrow F[x + 1])) \rightarrow (\forall x)\ F[x]$.

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \ : \ \{0, 1, +, \cdot, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ and $\cdot$ are binary functions.

(iii) $=$ is a binary predicate.

Its axiom set is the following:

($\mathcal{A}1.$) $(\forall x)\ \neg(x+1) = 0$.

($\mathcal{A}2.$) $(\forall x)(\forall y)\ [(x+1) = (y+1)] \rightarrow (x = y)$.

($\mathcal{A}3.$) $(F[0] \wedge (\forall x)\ (F[x] \rightarrow F[x+1])) \rightarrow (\forall x)\ F[x]$.

($\mathcal{A}4.$) $(\forall x)\ (x + 0 = x)$.

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \; : \; \{0, 1, +, \cdot, =\}$$

where,

(i) 0 and 1 are constants.
(ii) $+$ and $\cdot$ are binary functions.
(iii) $=$ is a binary predicate.

Its axiom set is the following:

($\mathcal{A}$1.) $(\forall x)\ \neg(x + 1) = 0$.
($\mathcal{A}$2.) $(\forall x)(\forall y)\ [(x + 1) = (y + 1)] \rightarrow (x = y)$.
($\mathcal{A}$3.) $(F[0] \wedge (\forall x)\ (F[x] \rightarrow F[x + 1])) \rightarrow (\forall x)\ F[x]$.
($\mathcal{A}$4.) $(\forall x)\ (x + 0 = x)$.
($\mathcal{A}$5.) $(\forall x)(\forall y)\ x + (y + 1) = (x + y) + 1$.

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \ : \ \{0, 1, +, \cdot, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ and $\cdot$ are binary functions.

(iii) $=$ is a binary predicate.

Its axiom set is the following:

($\mathcal{A}$1.) $(\forall x)\ \neg(x+1) = 0$.

($\mathcal{A}$2.) $(\forall x)(\forall y)\ [(x+1) = (y+1)] \rightarrow (x = y)$.

($\mathcal{A}$3.) $(F[0] \wedge (\forall x)\ (F[x] \rightarrow F[x+1])) \rightarrow (\forall x)\ F[x]$.

($\mathcal{A}$4.) $(\forall x)\ (x + 0 = x)$.

($\mathcal{A}$5.) $(\forall x)(\forall y)\ x + (y+1) = (x+y)+1$.

($\mathcal{A}$6.) $(\forall x)\ x \cdot 0 = 0$.

Peano Arithmetic (PA)

Main Issues

The theory of Peano Arithmetic (T_{PA}) or **first-order arithmetic** has the signature:

$$\Sigma_{PA} \ : \ \{0, 1, +, \cdot, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ and $\cdot$ are binary functions.

(iii) $=$ is a binary predicate.

Its axiom set is the following:

($\mathcal{A}1.$) $(\forall x)\ \neg(x + 1) = 0.$

($\mathcal{A}2.$) $(\forall x)(\forall y)\ [(x + 1) = (y + 1)] \rightarrow (x = y).$

($\mathcal{A}3.$) $(F[0] \wedge (\forall x)\ (F[x] \rightarrow F[x + 1])) \rightarrow (\forall x)\ F[x].$

($\mathcal{A}4.$) $(\forall x)\ (x + 0 = x).$

($\mathcal{A}5.$) $(\forall x)(\forall y)\ x + (y + 1) = (x + y) + 1.$

($\mathcal{A}6.$) $(\forall x)\ x \cdot 0 = 0.$

($\mathcal{A}7.$) $(\forall x)(\forall y)\ x \cdot (y + 1) = x \cdot y + x.$

Peano Arithmetic

Peano Arithmetic

Note

Induction (Axiom ($\mathcal{A}3.$)) an axiom schema.

Peano Arithmetic

Note

Induction (Axiom $(\mathcal{A}3.)$*) an axiom schema.*

Intended Interpretation

Peano Arithmetic

Note

Induction (Axiom $(\mathcal{A}3.)$*) an axiom schema.*

Intended Interpretation

Typically a theory has an intended interpretation

Peano Arithmetic

Note

Induction (Axiom $(\mathcal{A}3.)$*) an axiom schema.*

Intended Interpretation

Typically a theory has an intended interpretation (unless you have nothing better to do with your time than spinning theories).

T_{PA} has an intended interpretation, viz., the domain $\mathbb{N}$ and our regular (everyday arithmetic) interpretation of 0, 1, $+$, $\cdot$ and $=$.

Peano Arithmetic

Note

Induction (Axiom $(\mathcal{A}3.)$*) an axiom schema.*

Intended Interpretation

Typically a theory has an intended interpretation (unless you have nothing better to do with your time than spinning theories).

T_{PA} has an intended interpretation, viz., the domain $\mathbb{N}$ and our regular (everyday arithmetic) interpretation of 0, 1, $+$, $\cdot$ and $=$. We use the notation $T_{PA}(\mathbb{N})$ to indicate the fact that we are reasoning about formulae in the intended interpretation.

Peano Arithmetic

Note

Induction (Axiom $(\mathcal{A}3.)$*) an axiom schema.*

Intended Interpretation

Typically a theory has an intended interpretation (unless you have nothing better to do with your time than spinning theories).

T_{PA} has an intended interpretation, viz., the domain $\mathbb{N}$ and our regular (everyday arithmetic) interpretation of 0, 1, $+$, $\cdot$ and $=$. We use the notation $T_{PA}(\mathbb{N})$ to indicate the fact that we are reasoning about formulae in the intended interpretation. Observe that a formula F can be satisfiable in $T_{PA}(\mathbb{N})$, but not T_{PA}-valid.

Peano Arithmetic

Note

Induction (Axiom $(\mathcal{A}3.)$) *an axiom schema.*

Intended Interpretation

Typically a theory has an intended interpretation (unless you have nothing better to do with your time than spinning theories).

T_{PA} has an intended interpretation, viz., the domain $\mathbb{N}$ and our regular (everyday arithmetic) interpretation of 0, 1, $+$, $\cdot$ and $=$. We use the notation $T_{PA}(\mathbb{N})$ to indicate the fact that we are reasoning about formulae in the intended interpretation. Observe that a formula F can be satisfiable in $T_{PA}(\mathbb{N})$, but not T_{PA}-valid. Such a formula is called $T_{PA}(\mathbb{N})$-valid.

Peano Arithmetic

Note

Induction (Axiom $(\mathcal{A}3.)$*) an axiom schema.*

Intended Interpretation

Typically a theory has an intended interpretation (unless you have nothing better to do with your time than spinning theories).

T_{PA} has an intended interpretation, viz., the domain $\mathbb{N}$ and our regular (everyday arithmetic) interpretation of 0, 1, $+$, $\cdot$ and $=$. We use the notation $T_{PA}(\mathbb{N})$ to indicate the fact that we are reasoning about formulae in the intended interpretation. Observe that a formula F can be satisfiable in $T_{PA}(\mathbb{N})$, but not T_{PA}-valid. Such a formula is called $T_{PA}(\mathbb{N})$-valid.

Example

Is the following Σ_{PA} formula satisfiable?

Peano Arithmetic

Note

Induction (Axiom $(\mathcal{A}3.)$) *an axiom schema.*

Intended Interpretation

Typically a theory has an intended interpretation (unless you have nothing better to do with your time than spinning theories).

T_{PA} has an intended interpretation, viz., the domain $\mathbb{N}$ and our regular (everyday arithmetic) interpretation of 0, 1, $+$, $\cdot$ and $=$. We use the notation $T_{PA}(\mathbb{N})$ to indicate the fact that we are reasoning about formulae in the intended interpretation. Observe that a formula F can be satisfiable in $T_{PA}(\mathbb{N})$, but not T_{PA}-valid. Such a formula is called $T_{PA}(\mathbb{N})$-valid.

Example

Is the following Σ_{PA} formula satisfiable?

$$(\exists x)(\exists y)(\exists z)\ [x \neq 0 \land y \neq 0 \land x \cdot x + y \cdot y = z \cdot z]$$

Peano Arithmetic

Note

Induction (Axiom $(\mathcal{A}3.)$) *an axiom schema.*

Intended Interpretation

Typically a theory has an intended interpretation (unless you have nothing better to do with your time than spinning theories).

T_{PA} has an intended interpretation, viz., the domain $\mathbb{N}$ and our regular (everyday arithmetic) interpretation of 0, 1, $+$, $\cdot$ and $=$. We use the notation $T_{PA}(\mathbb{N})$ to indicate the fact that we are reasoning about formulae in the intended interpretation. Observe that a formula F can be satisfiable in $T_{PA}(\mathbb{N})$, but not T_{PA}-valid. Such a formula is called $T_{PA}(\mathbb{N})$-valid.

Example

Is the following Σ_{PA} formula satisfiable?

$$(\exists x)(\exists y)(\exists z)\ [x \neq 0 \wedge y \neq 0 \wedge x \cdot x + y \cdot y = z \cdot z]$$

Is it T_{PA}-valid?

Peano Arithmetic

Note

Induction (Axiom $(\mathcal{A}3.)$*) an axiom schema.*

Intended Interpretation

Typically a theory has an intended interpretation (unless you have nothing better to do with your time than spinning theories).

T_{PA} has an intended interpretation, viz., the domain $\mathbb{N}$ and our regular (everyday arithmetic) interpretation of 0, 1, $+$, $\cdot$ and $=$. We use the notation $T_{PA}(\mathbb{N})$ to indicate the fact that we are reasoning about formulae in the intended interpretation. Observe that a formula F can be satisfiable in $T_{PA}(\mathbb{N})$, but not T_{PA}-valid. Such a formula is called $T_{PA}(\mathbb{N})$-valid.

Example

Is the following Σ_{PA} formula satisfiable?

$$(\exists x)(\exists y)(\exists z)\ [x \neq 0 \land y \neq 0 \land x \cdot x + y \cdot y = z \cdot z]$$

Is it T_{PA}-valid? Is it $T_{PA}(\mathbb{N})$-valid?

Final observations

Final observations

Note

T_{PA} is undecidable.

Final observations

Note

T_{PA} is undecidable. Predicates such as $>$ and $\geq$ can be represented in T_{PA},

Final observations

Note

T_{PA} is undecidable. Predicates such as $>$ and $\geq$ can be represented in T_{PA}, i.e., additional axioms are not needed.

Final observations

Note

T_{PA} is undecidable. Predicates such as $>$ and $\geq$ can be represented in T_{PA}, i.e., additional axioms are not needed.

Gödel's first incompleteness theorem establishes that T_{PA} does not capture true arithmetic in that there exist closed formulae in T_{PA} that are valid propositions in number theory but are not provable in T_{PA}.

Presburger Arithmetic

Presburger Arithmetic

Main Issues

The theory of Presburger Arithmetic ($T_{\mathbb{N}}$) has the signature:

$$\Sigma_{\mathbb{N}} \;:\; \{0, 1, +, =\}$$

Presburger Arithmetic

Main Issues

The theory of Presburger Arithmetic ($T_{\mathbb{N}}$) has the signature:

$$\Sigma_{\mathbb{N}} \ : \ \{0, 1, +, =\}$$

where,

Presburger Arithmetic

Main Issues

The theory of Presburger Arithmetic ($T_{\mathbb{N}}$) has the signature:

$$\Sigma_{\mathbb{N}} \ : \ \{0, 1, +, =\}$$

where,

(i) 0 and 1 are constants.

Presburger Arithmetic

Main Issues

The theory of Presburger Arithmetic ($T_{\mathbb{N}}$) has the signature:

$$\Sigma_{\mathbb{N}} \ : \ \{0, 1, +, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ is a binary function.

Presburger Arithmetic

Main Issues

The theory of Presburger Arithmetic ($T_{\mathbb{N}}$) has the signature:

$$\Sigma_{\mathbb{N}} \;:\; \{0, 1, +, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ is a binary function.

(iii) $=$ is a binary predicate.

Presburger Arithmetic

Main Issues

The theory of Presburger Arithmetic ($T_{\mathbb{N}}$) has the signature:

$$\Sigma_{\mathbb{N}} \ : \ \{0, 1, +, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ is a binary function.

(iii) $=$ is a binary predicate.

Its axiom set (subset of T_{PA}) is the following:

Presburger Arithmetic

Main Issues

The theory of Presburger Arithmetic ($T_{\mathbb{N}}$) has the signature:

$$\Sigma_{\mathbb{N}} \ : \ \{0, 1, +, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ is a binary function.

(iii) $=$ is a binary predicate.

Its axiom set (subset of T_{PA}) is the following:

($\mathcal{A}$1.) $(\forall x)\ \neg(x + 1) = 0$.

Presburger Arithmetic

Main Issues

The theory of Presburger Arithmetic ($T_{\mathbb{N}}$) has the signature:

$$\Sigma_{\mathbb{N}} \ : \ \{0, 1, +, =\}$$

where,

(i) 0 and 1 are constants.
(ii) $+$ is a binary function.
(iii) $=$ is a binary predicate.

Its axiom set (subset of T_{PA}) is the following:

($\mathcal{A}1.$) $(\forall x)\ \neg(x + 1) = 0.$

($\mathcal{A}2.$) $(\forall x)(\forall y)\ [(x + 1) = (y + 1)] \rightarrow (x = y).$

Presburger Arithmetic

Main Issues

The theory of Presburger Arithmetic ($T_{\mathbb{N}}$) has the signature:

$$\Sigma_{\mathbb{N}} \ : \ \{0, 1, +, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ is a binary function.

(iii) $=$ is a binary predicate.

Its axiom set (subset of T_{PA}) is the following:

($\mathcal{A}$1.) $(\forall x)\ \neg(x + 1) = 0$.

($\mathcal{A}$2.) $(\forall x)(\forall y)\ [(x + 1) = (y + 1)] \rightarrow (x = y)$.

($\mathcal{A}$3.) $(F[0] \land (\forall x)\ (F[x] \rightarrow F[x + 1])) \rightarrow (\forall x)\ F[x]$.

Presburger Arithmetic

Main Issues

The theory of Presburger Arithmetic ($T_{\mathbb{N}}$) has the signature:

$$\Sigma_{\mathbb{N}} \; : \; \{0, 1, +, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ is a binary function.

(iii) $=$ is a binary predicate.

Its axiom set (subset of T_{PA}) is the following:

($\mathcal{A}1.$) $(\forall x)\ \neg(x + 1) = 0$.

($\mathcal{A}2.$) $(\forall x)(\forall y)\ [(x + 1) = (y + 1)] \rightarrow (x = y)$.

($\mathcal{A}3.$) $(F[0] \wedge (\forall x)\ (F[x] \rightarrow F[x + 1])) \rightarrow (\forall x)\ F[x]$.

($\mathcal{A}4.$) $(\forall x)\ (x + 0 = x)$.

Presburger Arithmetic

Main Issues

The theory of Presburger Arithmetic ($T_{\mathbb{N}}$) has the signature:

$$\Sigma_{\mathbb{N}} \;:\; \{0, 1, +, =\}$$

where,

(i) 0 and 1 are constants.

(ii) $+$ is a binary function.

(iii) $=$ is a binary predicate.

Its axiom set (subset of T_{PA}) is the following:

($\mathcal{A}1.$) $(\forall x)\ \neg(x+1) = 0.$

($\mathcal{A}2.$) $(\forall x)(\forall y)\ [(x+1) = (y+1)] \rightarrow (x = y).$

($\mathcal{A}3.$) $(F[0] \wedge (\forall x)\ (F[x] \rightarrow F[x+1])) \rightarrow (\forall x)\ F[x].$

($\mathcal{A}4.$) $(\forall x)\ (x + 0 = x).$

($\mathcal{A}5.$) $(\forall x)(\forall y)\ x + (y+1) = (x+y) + 1.$

Additional issues

Additional issues

Points

Additional issues

Points

(i) Induction is still a schema.

Additional issues

Points

(i) Induction is still a schema.

(ii) Intended interpretation is everyday arithmetic.

Additional issues

Points

(i) Induction is still a schema.

(ii) Intended interpretation is everyday arithmetic. We use $T_{\mathbb{N}}(\mathbb{N})$ to reason in the intended interpretation.

Additional issues

Points

(i) Induction is still a schema.

(ii) Intended interpretation is everyday arithmetic. We use $T_{\mathbb{N}}(\mathbb{N})$ to reason in the intended interpretation.

(iii) Can be used to represent all integers.

Additional issues

Points

(i) Induction is still a schema.

(ii) Intended interpretation is everyday arithmetic. We use $T_{\mathbb{N}}(\mathbb{N})$ to reason in the intended interpretation.

(iii) Can be used to represent all integers. e.g. $(\exists z)\ z + 7 = 5$, where z is intended to range over $\mathbb{Z}$.

(iv) Subtraction and strict inequality can be handled.

Additional issues

Points

(i) Induction is still a schema.

(ii) Intended interpretation is everyday arithmetic. We use $T_{\mathbb{N}}(\mathbb{N})$ to reason in the intended interpretation.

(iii) Can be used to represent all integers. e.g. $(\exists z)\ z + 7 = 5$, where z is intended to range over $\mathbb{Z}$.

(iv) Subtraction and strict inequality can be handled.

(v) $T_{\mathbb{N}}$ is decidable.

Additional issues

Points

(i) Induction is still a schema.

(ii) Intended interpretation is everyday arithmetic. We use $T_{\mathbb{N}}(\mathbb{N})$ to reason in the intended interpretation.

(iii) Can be used to represent all integers. e.g. $(\exists z)\ z + 7 = 5$, where z is intended to range over $\mathbb{Z}$.

(iv) Subtraction and strict inequality can be handled.

(v) $T_{\mathbb{N}}$ is decidable.

Example

Is the following formula representable in $T_{\mathbb{N}}$?

Additional issues

Points

(i) Induction is still a schema.

(ii) Intended interpretation is everyday arithmetic. We use $T_{\mathbb{N}}(\mathbb{N})$ to reason in the intended interpretation.

(iii) Can be used to represent all integers. e.g. $(\exists z)\ z + 7 = 5$, where z is intended to range over $\mathbb{Z}$.

(iv) Subtraction and strict inequality can be handled.

(v) $T_{\mathbb{N}}$ is decidable.

Example

Is the following formula representable in $T_{\mathbb{N}}$?

$$(\forall x)(\forall w)(\exists y)(\exists z)\ [x + 2 \cdot y - z - 13 > -3 \cdot w + 5]$$

Integer Arithmetic

Integer Arithmetic

Main issues

Integer Arithmetic

Main issues

The theory of integers, $T_{\mathbb{Z}}$, has signature

$$\Sigma_{\mathbb{Z}} \ : \ \{- \ldots, -2, -1, 0, 1, 2, \ldots, +, -, =, >\}$$

Integer Arithmetic

Main issues

The theory of integers, $T_{\mathbb{Z}}$, has signature

$$\Sigma_{\mathbb{Z}} \; : \; \{- \ldots, -2, -1, 0, 1, 2, \ldots, +, -, =, >\}$$

where,

Integer Arithmetic

Main issues

The theory of integers, $T_{\mathbb{Z}}$, has signature

$$\Sigma_{\mathbb{Z}} \ : \ \{- \ldots, -2, -1, 0, 1, 2, \ldots, +, -, =, >\}$$

where,

(i) The integers are constants intended to be assigned to the obvious values in the intended domain of integers $\mathbb{Z}$.

Integer Arithmetic

Main issues

The theory of integers, $T_{\mathbb{Z}}$, has signature

$$\Sigma_{\mathbb{Z}} \; : \; \{- \ldots, -2, -1, 0, 1, 2, \ldots, +, -, =, >\}$$

where,

(i) The integers are constants intended to be assigned to the obvious values in the intended domain of integers $\mathbb{Z}$.

(ii) $+$ and $-$ are intended to represent the corresponding functions in $\mathbb{Z}$.

Integer Arithmetic

Main issues

The theory of integers, $T_{\mathbb{Z}}$, has signature

$$\Sigma_{\mathbb{Z}} \ : \ \{- \ldots, -2, -1, 0, 1, 2, \ldots, +, -, =, >\}$$

where,

(i) The integers are constants intended to be assigned to the obvious values in the intended domain of integers $\mathbb{Z}$.

(ii) $+$ and $-$ are intended to represent the corresponding functions in $\mathbb{Z}$.

(iii) $=$ and $>$ are binary predicates intended to represent the corresponding predicates over $\mathbb{Z}$.

Integer Arithmetic

Main issues

The theory of integers, $T_{\mathbb{Z}}$, has signature

$$\Sigma_{\mathbb{Z}} \; : \; \{-\ldots, -2, -1, 0, 1, 2, \ldots, +, -, =, >\}$$

where,

(i) The integers are constants intended to be assigned to the obvious values in the intended domain of integers $\mathbb{Z}$.

(ii) $+$ and $-$ are intended to represent the corresponding functions in $\mathbb{Z}$.

(iii) $=$ and $>$ are binary predicates intended to represent the corresponding predicates over $\mathbb{Z}$.

The theory of integers in the intended interpretation will be denoted by $T_{\mathbb{Z}}(\mathbb{Z})$.

Integer Arithmetic

Main issues

The theory of integers, $T_{\mathbb{Z}}$, has signature

$$\Sigma_{\mathbb{Z}} \ : \ \{-\ldots, -2, -1, 0, 1, 2, \ldots, +, -, =, >\}$$

where,

(i) The integers are constants intended to be assigned to the obvious values in the intended domain of integers $\mathbb{Z}$.

(ii) $+$ and $-$ are intended to represent the corresponding functions in $\mathbb{Z}$.

(iii) $=$ and $>$ are binary predicates intended to represent the corresponding predicates over $\mathbb{Z}$.

The theory of integers in the intended interpretation will be denoted by $T_{\mathbb{Z}}(\mathbb{Z})$.

Note

Integer Arithmetic

Main issues

The theory of integers, $T_{\mathbb{Z}}$, has signature

$$\Sigma_{\mathbb{Z}} \ : \ \{-\ldots, -2, -1, 0, 1, 2, \ldots, +, -, =, >\}$$

where,

(i) The integers are constants intended to be assigned to the obvious values in the intended domain of integers $\mathbb{Z}$.

(ii) $+$ and $-$ are intended to represent the corresponding functions in $\mathbb{Z}$.

(iii) $=$ and $>$ are binary predicates intended to represent the corresponding predicates over $\mathbb{Z}$.

The theory of integers in the intended interpretation will be denoted by $T_{\mathbb{Z}}(\mathbb{Z})$.

Note

(i) *Every formula in $T_{\mathbb{Z}}$ can be reduced to $T_{\mathbb{N}}$, and hence a separate axiomatization is unnecessary.*

Integer Arithmetic

Main issues

The theory of integers, $T_{\mathbb{Z}}$, has signature

$$\Sigma_{\mathbb{Z}} \; : \; \{- \ldots, -2, -1, 0, 1, 2, \ldots, +, -, =, >\}$$

where,

(i) The integers are constants intended to be assigned to the obvious values in the intended domain of integers $\mathbb{Z}$.

(ii) $+$ and $-$ are intended to represent the corresponding functions in $\mathbb{Z}$.

(iii) $=$ and $>$ are binary predicates intended to represent the corresponding predicates over $\mathbb{Z}$.

The theory of integers in the intended interpretation will be denoted by $T_{\mathbb{Z}}(\mathbb{Z})$.

Note

(i) *Every formula in $T_{\mathbb{Z}}$ can be reduced to $T_{\mathbb{N}}$, and hence a separate axiomatization is unnecessary.*

(ii) *$T_{\mathbb{Z}}$ is merely a more convenient form to reason about integers.*

Integer Arithmetic

Main issues

The theory of integers, $T_{\mathbb{Z}}$, has signature

$$\Sigma_{\mathbb{Z}} \ : \ \{-\ldots, -2, -1, 0, 1, 2, \ldots, +, -, =, >\}$$

where,

(i) The integers are constants intended to be assigned to the obvious values in the intended domain of integers $\mathbb{Z}$.

(ii) $+$ and $-$ are intended to represent the corresponding functions in $\mathbb{Z}$.

(iii) $=$ and $>$ are binary predicates intended to represent the corresponding predicates over $\mathbb{Z}$.

The theory of integers in the intended interpretation will be denoted by $T_{\mathbb{Z}}(\mathbb{Z})$.

Note

(i) *Every formula in $T_{\mathbb{Z}}$ can be reduced to $T_{\mathbb{N}}$, and hence a separate axiomatization is unnecessary.*

(ii) *$T_{\mathbb{Z}}$ is merely a more convenient form to reason about integers.*

(iii) *$T_{\mathbb{Z}}$ itself is decidable.*

Examples

Examples

Example

Show that the following formulae are $T_{\mathbb{Z}}(\mathbb{Z})$-valid.

Examples

Example

Show that the following formulae are $T_{\mathbb{Z}}(\mathbb{Z})$-valid.

1. $$F \; : \; (\forall x)(\forall y)(\forall z) \; [(x > z) \land (y \geq 0)] \rightarrow [(x + y) > z]$$

Examples

Example

Show that the following formulae are $T_{\mathbb{Z}}(\mathbb{Z})$-valid.

1. $$F \; : \; (\forall x)(\forall y)(\forall z)\ [(x > z) \wedge (y \geq 0)] \rightarrow [(x + y) > z]$$

2. $$G \; : \; (\forall x)(\forall y)\ [(x > 0) \wedge (x = 2 \cdot y \vee x = 2 \cdot y + 1)] \rightarrow [(x - y) > 0]$$

Examples

Example

Show that the following formulae are $T_{\mathbb{Z}}(\mathbb{Z})$-valid.

1. $$F \; : \; (\forall x)(\forall y)(\forall z)\; [(x > z) \wedge (y \geq 0)] \rightarrow [(x + y) > z]$$

2. $$G \; : \; (\forall x)(\forall y)\; [(x > 0) \wedge (x = 2 \cdot y \vee x = 2 \cdot y + 1)] \rightarrow [(x - y) > 0]$$

Are they $T_{\mathbb{Z}}$-valid?