First Order Theories - Rationals and Reals

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Outline

1 Introduction

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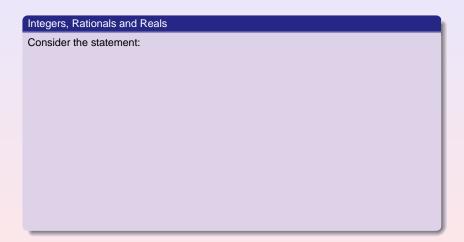
1 Introduction

Theory of Rationals

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- 2 Theory of Rationals
- Theory of Reals



Integers, Rationals and Reals

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Main points	

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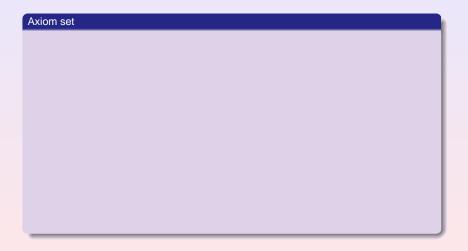
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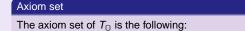
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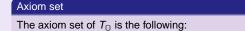
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- (ii) + is a binary function.
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- (iv) = and \geq are binary predicates.







Axiom set

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$$(\forall x) \ n \cdot x = 0 \rightarrow x = 0$$
, for all positive integers n (torsion-free).

Axiom set

$$(A1.)$$
 $(\forall x)(\forall y)$ $[(x > y) \land (y > x)] \rightarrow (y = x).$

$$(\mathcal{A}2.)$$
 $(\forall x)(\forall y)(\forall z)$ $[(x \geq y) \land (y \geq z)] \rightarrow (x \geq z).$

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The axiom set of $T_{\mathbb{O}}$ is the following:

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Why cannot *n* be an arbitrary rational in Axioms ($\mathcal{A}9$.) and (\mathcal{A} .10)?



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$$(\mathcal{A}8.) \ (\forall x)(\forall y)(\forall z) \ (x \geq y) \rightarrow (x+z) \geq (y+z).$$

(A9.)
$$(\forall x)(\forall y)(\forall z) (x \cdot y) \cdot z = x \cdot (y \cdot z)$$
.

$$(\mathcal{A}10.) \ (\forall x) \ 1 \cdot x = 1.$$

$$(\mathcal{A}11.) \ (\forall x) \ (x \neq 0) \rightarrow (\exists y) \ x \cdot y = 1.$$

$$(\mathcal{A}12.) \ (\forall x)(\forall y) \ x \cdot y = y \cdot x.$$

$$(\mathcal{A}13.) \ (\forall x)(\forall y) \ x > 0 \land y > 0 \rightarrow x \cdot y > 0.$$

Axiom set		

$$(\mathcal{A}14.) \ (\forall x)(\forall y)(\forall z) \ x \cdot (y+z) = x \cdot y + x \cdot z).$$

$$(\mathcal{A}14.) \ (\forall x)(\forall y)(\forall z) \ x \cdot (y+z) = x \cdot y + x \cdot z).$$

$$(A15.) 0 \neq 1.$$

$$(\mathcal{A}14.) \ (\forall x)(\forall y)(\forall z) \ x \cdot (y+z) = x \cdot y + x \cdot z).$$

$$(A15.) 0 \neq 1.$$

$$(\mathcal{A}16.) \ (\forall x)(\exists y) \ x = y^2 \lor -x = y^2.$$

$$(\mathcal{A}14.) \ (\forall x)(\forall y)(\forall z) \ x \cdot (y+z) = x \cdot y + x \cdot z).$$

$$(A15.) 0 \neq 1.$$

$$(\mathcal{A}16.) \ (\forall x)(\exists y) \ x = y^2 \lor -x = y^2.$$

(A17.) For each odd integer
$$n$$
, $(\forall \overline{x})(\exists y)[y^n+x_1\cdot y^{n-1}+\ldots x_{n-1}\cdot y+x_n=0]$,

$$(\mathcal{A}14.) \ (\forall x)(\forall y)(\forall z) \ x \cdot (y+z) = x \cdot y + x \cdot z).$$

$$(A15.) 0 \neq 1.$$

$$(A16.) \ (\forall x)(\exists y) \ x = y^2 \lor -x = y^2.$$

(A17.) For each odd integer
$$n$$
, $(\forall \overline{x})(\exists y)[y^n+x_1\cdot y^{n-1}+\ldots x_{n-1}\cdot y+x_n=0]$, where $\overline{x}=[x_1,x_2,\ldots x_n]$.