First Order Theories - Recursive Data Structures

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Recursive data structures

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Theory of Lists

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Main points		

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- (v) = is a binary predicate.

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The axiom set of T_{cons} is the following:

($\mathcal{A}1$.) The axioms of reflexivity, symmetry and transitivity of T_E .

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- (A2.) Instantiations of the function congruence schema for cons, car, and, cdr.

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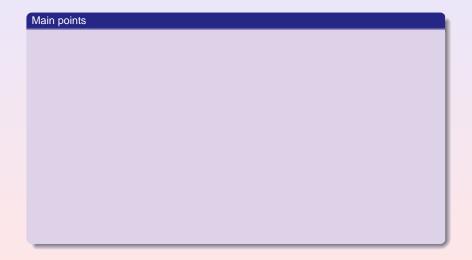
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- $(\mathcal{A}7.)$ $(\forall x)(\forall y) \neg atom(cons(x, y)).$

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- (ii) n projection functions $\pi_1^C, \pi_2^C, \dots, \pi_n^C$.

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- (iii) One atom predicate atom_C.

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Associated with each RDS is an instantiation of the following axiom schema:

($\mathcal{A}1$.) The axioms of reflexivity, symmetry and transitivity of T_E .

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- (iii) One atom predicate $atom_C$.

- ($\mathcal{A}1$.) The axioms of reflexivity, symmetry and transitivity of T_E .
- (A2.) Instantiations of the function congruence axiom schema for the constructor C and the set of projectors $\pi_1^C, \pi_2^C, \dots, \pi_n^C$.

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- ($\mathcal{A}2$.) Instantiations of the function congruence axiom schema for the constructor C and the set of projectors $\pi_1^C, \pi_2^C, \dots, \pi_n^C$.
- (A3.) An instantiation of the predicate congruence axiom schema for atom_C.

$$(\mathcal{A}4.) \ (\forall x_1)(\forall x_2)...(\forall x_n) \ \pi_i^{C}(C(x_1, x_2, ..., x_n)) = x_i \text{ for each } i \in \{1, 2, ..., n\}.$$

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- (.42.) Instantiations of the function congruence axiom schema for the constructor C and the set of projectors $\pi_1^C, \pi_2^C, \dots, \pi_n^C$.
- ($\mathcal{A}3$.) An instantiation of the predicate congruence axiom schema for $atom_{\mathbb{C}}$.
- (A4.) $(\forall x_1)(\forall x_2)...(\forall x_n) \pi_i^{\mathbb{C}}(C(x_1, x_2,...,x_n)) = x_i \text{ for each } i \in \{1, 2,...,n\}.$
- $(\mathcal{A}5.) \ (\forall x) \ \neg \text{atom}_{\mathbb{C}}(x) \rightarrow \mathcal{C}(\pi_1^{\mathbb{C}}(x), \pi_2^{\mathbb{C}}(x), \dots, \pi_n^{\mathbb{C}}(x)) = x.$

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$$(\mathcal{A}4.) \ (\forall x_1)(\forall x_2) \dots (\forall x_n) \ \pi_i^{\mathbb{C}}(\mathbb{C}(x_1, x_2, \dots, x_n)) = x_i \text{ for each } i \in \{1, 2, \dots, n\}.$$

(A5.)
$$(\forall x) \neg atom_C(x) \to C(\pi_1^C(x), \pi_2^C(x), \dots, \pi_n^C(x)) = x.$$

$$(\mathcal{A}6.)$$
 $(\forall x_1)(\forall x_2)...(\forall x_n) \neg atom_C(C(x_1, x_2,...,x_n)).$

Acylic lists

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 - $(\mathcal{A}3.)$ $(\forall x)$ car $(car(x)) \neq x$.

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 - (A1.) $(\forall x) \operatorname{car}(x) \neq x$.
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 - (A3.) $(\forall x)$ car(car(x)) $\neq x$.
 - $(\mathcal{A}4.)$ $(\forall x)$ car(cdr(x)) $\neq x$.

- (i) The theory of acyclic lists, $T_{\rm cms}^+$, is used to reason about structures such as stacks, which are naturally acyclic.
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```
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(A3.)
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$$(\forall x) \operatorname{car}(\operatorname{cdr}(x)) \neq x$$
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```

$$(\mathcal{A}4.)$$
 $(\forall x)$ car(cdr(x)) $\neq x$.
 $(\mathcal{A}5.)$ $(\forall x)$ cdr(car(x)) $\neq x$.

$$(A6.)$$

Specifying atomic behavior

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The axioms of $T_{\rm cons}^+$ do not specify the behavior of cons and cdr on atoms.

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The axioms of $T_{\rm cons}^+$ do not specify the behavior of cons and cdr on atoms. Adding the axiom,

$$(\forall x) \operatorname{atom}(x) \rightarrow [\operatorname{atom}(\operatorname{car}(x)) \wedge \operatorname{atom}(\operatorname{cdr}(x))]$$

gives a new theory, viz., $T_{\rm cons}^{\rm atom}$.

Theory of Lists with Equality

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Example

Theory of Lists with Equality

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$$F \ : \ [(\operatorname{car}(a) = \operatorname{car}(b)) \land (\operatorname{cdr}(a) = \operatorname{cdr}(b)) \land \neg \operatorname{atom}(a) \land \neg \operatorname{atom}(b)] \rightarrow [f(a) = f(b)]$$

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Is F $T_{cons}^{=}$ -valid? Is it T_{cons} -valid?