First Order Theories - Recursive Data Structures

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March 8 2013

Outline

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(i) cons is a binary function called the constructor: $\text{cons}(a, b)$ is the list obtained by concatenating a to b.

(ii) car is a unary function, called the left projector: $\text{car}(\text{cons}(a, b)) = a$.

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The axioms of T_{cons}^{+} do not specify the behavior of cons and cdr on atoms. Adding the axiom,

$$(\forall x)\ \text{atom}(x) \rightarrow [\text{atom}(\text{car}(x)) \wedge \text{atom}(\text{cdr}(x))]$$

gives a new theory, viz., $T_{\text{cons}}^{\text{atom}}$.

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