Induction - Stepwise Induction

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Outline

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Axiom Schema (for $\mathbb{N}$)

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Note

(i) *Showing that $P(0)$ is* **true** *is called the basis step.*

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Note

(i) *Showing that $P(0)$ is* **true** *is called the basis step.*

(ii) *The assumption that $P(k)$ is* **true***, is called the inductive hypothesis.*

Stepwise Induction on Integral domains

Example

Show that the sum of the first n integers is $\frac{n \cdot (n+1)}{2}$.

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BASIS ($P(1)$):

$$LHS = \sum_{i=1}^{1} i$$

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Thus, $LHS = RHS$ and $P(1)$ is true. □

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Let us assume that $P(k)$ is true, i.e., assume that

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Since, *LHS*=*RHS*, we have shown that $P(k) \rightarrow P(k+1)$.

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Applying the first principle of mathematical induction, we conclude that the conjecture is true. □

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Main Ideas

(i) Mathematicize the conjecture.
(ii) Prove the basis (usually $P(1)$ and usually easy.)
(iii) Assume $P(k)$.
(iv) Show $P(k+1)$. (The hard part. Use mathematical manipulation.)

Another Stepwise Induction Example

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Show that the sum of the squares of the first n integers is $\frac{n \cdot (n+1) \cdot (2n+1)}{6}$,

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Thus, $LHS = RHS$ and $P(1)$ is true. □

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&= \frac{k+1}{6}(k \cdot (2k+1) + 6 \cdot (k+1)) \\
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□

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&= \frac{(k+1)\cdot(k+2)\cdot(2\cdot(k+1)+1)}{6}
\end{aligned}
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Completing the proof

Proof.

$$
\begin{aligned}
&= \frac{k+1}{6}(2k^2 + 4k + 3k + 6) \\
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Since, *LHS*=*RHS*, we have shown that $P(k) \longrightarrow P(k+1)$.

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Applying the first principle of mathematical induction, we conclude that the conjecture is true. □

Stepwise Induction on Lists

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Axiom Schema

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$$[((\forall u)\ \mathrm{atom}(u) \rightarrow F[u]) \wedge (\forall u)(\forall v)\ F[v] \rightarrow F[\mathrm{cons}(u, v)]] \rightarrow (\forall x)\ F[x].$$

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The theory T^{+}_{cons}

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Consider the theory T^{+}_{cons}, which is the theory T_{cons} augmented by the following axioms:

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Stepwise Induction on Lists

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$\mathcal{A}4.$ $(\forall x)(\forall y)\ \text{rvs}(\text{concat}(x, y)) = \text{concat}(\text{rvs}(y), \text{rvs}(x))$.

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$\mathcal{A}6.$ $(\forall u)(\forall v)\ \text{flat}(\text{cons}(u,v)) \leftrightarrow \text{atom}(u) \wedge \text{flat}(v).$

Example

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Prove that

$$(\forall x)\ \text{flat}(x) \rightarrow \text{rvs}(\text{rvs}(x) = x).$$