Induction - Well-Founded Induction

K. Subramani[1]

[1] Lane Department of Computer Science and Electrical Engineering
West Virginia University

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Outline

1. Well-Founded Induction

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A binary predicate $\prec$ over a set S, is a well-founded relation, if and only if there does not exist an infinite sequence $s_1, s_2, \ldots$ of elements of S, such that

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$$[(\forall n)\ (\forall n')\ ((n' \prec n) \rightarrow F(n')) \rightarrow F(n)] \rightarrow (\forall x)\ F(x).$$

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Axiom schema

$$((\forall \bar{n})\ ((\forall \bar{n'})\ (n' \prec n) \rightarrow F[\bar{n'}]) \rightarrow F[\bar{n}]) \rightarrow (\forall \bar{x})\ F[\bar{x}].$$

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Note

The only difference between lexicographic induction and well-formed induction, is that in the former we consider tuples rather than single elements.

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Compute $A(0, 0)$, $A(1, 1)$, $A(2, 2)$, $A(3, 3)$, $A(4, 4)$.

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Compute $A(0,0)$, $A(1,1)$, $A(2,2)$, $A(3,3)$, $A(4,4)$.
Define the relation $<_2$ for 2-tuples.
Argue that $(\forall x)(\forall y)\ A(x, y) > y$.

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Have we covered all the cases? Does the above process always halt?