Propositional Logic - Advanced

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16 January and 18 January, 2013

Outline

1. Review

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Review

Main Issues

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1. Reasons for logic.

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2. Propositions.

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3. Connectives

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Review

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1. Reasons for logic.
2. Propositions.
3. Connectives ($\perp$ and $\top$).
4. Semantics and interpretation.

Formula evaluation

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Under a given assignment I, a formula F evaluates to either **true** or **false**.

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Under a given assignment I, a formula F evaluates to either **true** or **false**. If it is the former, we say $I \models F$; if it is the latter, we say $I \not\models F$.

Checking whether an interpretation I models F

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Inductive Definition

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1. $I \models \top$.

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1. $I \models \top$.
2. $I \not\models \bot$.

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Inductive Definition

1. $I \models \top$.
2. $I \not\models \bot$.
3. $I \models P$ iff $I[P] = \textbf{true}$.

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Checking whether an interpretation I models F

Inductive Definition

1. $I \models \top$.
2. $I \not\models \bot$.
3. $I \models P$ iff $I[P] = \textbf{true}$.
4. $I \not\models P$ iff $I[P] = \textbf{false}$.
5. $I \models \neg P$ iff $I \not\models P$.

Checking whether an interpretation I models F

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1. $I \models \top$.
2. $I \not\models \bot$.
3. $I \models P$ iff $I[P] =$ **true**.
4. $I \not\models P$ iff $I[P] =$ **false**.
5. $I \models \neg P$ iff $I \not\models P$.
6. $I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$.

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6. $I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$.
7. $I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$.

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7. $I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$.
8. $I \models F_1 \rightarrow F_2$ iff if $I \models F_1$, then $I \models F_2$.

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Note

For implication, it is more convenient to use the definition:
$I \not\models F_1 \rightarrow F_2$ *iff* $I \models F_1$ *and* $I \not\models F_2$.

Example

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What does $F\ :\ (P \wedge Q) \rightarrow (P \vee \neg Q)$ evaluate to, under the interpretation $I\ \{P \rightarrow \textbf{true},\ Q \rightarrow \textbf{false}\}$?

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$A\vee)B$ is not a wff.

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(i) parentheses.
(ii) negation.
(iii) conjunction, disjunction.
(i) implication.
(ii) equivalence.

Use brackets and forget about precedence!

Outline

Tautologies

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A wff which is always **true** is called a *tautology*, while a wff which is always **false** is called a *contradiction*.

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$A \rightarrow A$.

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If A and B are two wffs, and $A \leftrightarrow B$ is a tautology, then A and B are said to be **equivalent wffs** (denoted by $A \Leftrightarrow B$) and can be substituted for one another.

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Example

$(A \rightarrow B) \Leftrightarrow (B' \rightarrow A')$

Common Tautological Equivalences

De Morgan's Laws

$$(A \vee B)' \Leftrightarrow (A' \wedge B')$$

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$(A \vee B)' \Leftrightarrow (A' \wedge B')$ $\qquad$ $(A \wedge B)' \Leftrightarrow (A' \vee B')$

Commutativity

$(A \vee B) \Leftrightarrow (B \vee A)$

Common Tautological Equivalences

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Commutativity

$(A \vee B) \Leftrightarrow (B \vee A)$ $\quad$ $(A \wedge B) \Leftrightarrow (B \wedge A)$

Associativity

$(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$

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Associativity

$(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$ $\quad$ $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$

Distributivity

$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$

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Associativity

$(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$ $\quad$ $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$

Distributivity

$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$ $\quad$ $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$

Exercise

Prove all the above tautologies, using reasoning.

Outline

1. Review
2. Interpretations
   - Tautologies
3. Propositional Logic
   - Valid Arguments
4. Checking validity
   - Truth Table Method
   - Intuitive Argument
   - Derivation Rule Method
   - Semantic Argument Method
5. Equivalence and Implication

Arguments

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An argument is a statement of the form:

$$(P_1 \wedge P_2 \wedge \ldots P_n) \to Q$$

where each of the P_is and Q are propositions.

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An argument is a statement of the form:

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Semantics

When can Q be logically deduced from $P_1, P_2, \ldots, P_n$?

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Semantics

When can Q be logically deduced from $P_1, P_2, \ldots, P_n$?

Example

$2 + 2 = 4$ and $7 + 3 = 10$. Therefore, a minute has 60 seconds. Is this valid?

Valid Arguments

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The argument

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The validity of an argument is based purely on its intrinsic structure and not on the specific meanings attached to its constituent propositions.

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Example

If John is hungry, he will eat. John is hungry. Therefore, he will eat.

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Note

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Example

If John is hungry, he will eat. John is hungry. Therefore, he will eat.
Symbolically, $(H \rightarrow E) \wedge H \rightarrow E$.

Outline

Truth Table Method

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Truth-table Method

Simply check if all rows of the truth-table are **true**.

Truth Table Method

Truth-table Method

Simply check if all rows of the truth-table are **true**. Horribly expensive!

Outline

Intuitive Argument Method

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Example

Is the argument

$$[(A \to B) \land (B \to C)] \to (A \to C)$$

valid?

Intuitive Argument Method

Example

Is the argument

$$[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$$

valid?
A is either **true** or **false**.

Intuitive Argument Method

Example

Is the argument

$$[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$$

valid?

A is either **true** or **false**. If *A* is **false**,

Intuitive Argument Method

Example

Is the argument

$$[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$$

valid?
A is either **true** or **false**. If A is **false**, the consequent is **true** and hence the entire argument is **true**.
If A is **true**, the argument reduces to $[B \wedge (B \rightarrow C)] \rightarrow C$.

Intuitive Argument Method

Example

Is the argument

$$[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$$

valid?
A is either **true** or **false**. If A is **false**, the consequent is **true** and hence the entire argument is **true**.
If A is **true**, the argument reduces to $[B \wedge (B \rightarrow C)] \rightarrow C$. Now B is either **true** or **false**...

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Derivation Rule Method

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Derivation Rules

We will use a set of derivation rules and manipulate the hypotheses to arrive at the desired conclusion.

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Proof Sequence

A proof sequence is a sequence of wffs in which each wff is either a hypothesis or the result of applying one of the formal system's derivation rules to earlier wffs in the sequence.

Derivation Rules

Rule Types

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Rule Types

(i) Equivalence Rules.

Derivation Rules

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(i) Equivalence Rules.
(ii) Inference Rules.

Derivation Rules

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Equivalence Rules

Expression	Equivalent to	Name of Rule

Derivation Rules

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Expression	Equivalent to	Name of Rule
$P \vee Q$	$Q \vee P$	Commutative - comm

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$P \vee Q$	$Q \vee P$	Commutative - comm
$P \wedge Q$	$Q \wedge P$	

Derivation Rules

Rule Types

(i) Equivalence Rules.

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Equivalence Rules

Expression	Equivalent to	Name of Rule
$P \vee Q$ $P \wedge Q$	$Q \vee P$ $Q \wedge P$	Commutative - comm
$P \vee (Q \vee R)$ $P \wedge (Q \wedge R)$	$(P \vee Q) \vee R$ $(P \wedge Q) \wedge R$	Associative -ass

Derivation Rules

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Equivalence Rules

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$P \vee Q$ $P \wedge Q$	$Q \vee P$ $Q \wedge P$	Commutative - comm
$P \vee (Q \vee R)$ $P \wedge (Q \wedge R)$	$(P \vee Q) \vee R$ $(P \wedge Q) \wedge R$	Associative -ass
$(P \vee Q)'$ $(P \wedge Q)'$	$P' \wedge Q'$ $P' \vee Q'$	De Morgan

Derivation Rules

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$P \vee Q$ $P \wedge Q$	$Q \vee P$ $Q \wedge P$	Commutative - comm
$P \vee (Q \vee R)$ $P \wedge (Q \wedge R)$	$(P \vee Q) \vee R$ $(P \wedge Q) \wedge R$	Associative -ass
$(P \vee Q)'$ $(P \wedge Q)'$	$P' \wedge Q'$ $P' \vee Q'$	De Morgan
$P \rightarrow Q$	$P' \vee Q$	Implication - imp

Derivation Rules

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Expression	Equivalent to	Name of Rule
$P \vee Q$ $P \wedge Q$	$Q \vee P$ $Q \wedge P$	Commutative - comm
$P \vee (Q \vee R)$ $P \wedge (Q \wedge R)$	$(P \vee Q) \vee R$ $(P \wedge Q) \wedge R$	Associative -ass
$(P \vee Q)'$ $(P \wedge Q)'$	$P' \wedge Q'$ $P' \vee Q'$	De Morgan
$P \rightarrow Q$	$P' \vee Q$	Implication - imp
P	$(P')'$	Double negation - dn

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Expression	Equivalent to	Name of Rule
$P \vee Q$ $P \wedge Q$	$Q \vee P$ $Q \wedge P$	Commutative - comm
$P \vee (Q \vee R)$ $P \wedge (Q \wedge R)$	$(P \vee Q) \vee R$ $(P \wedge Q) \wedge R$	Associative -ass
$(P \vee Q)'$ $(P \wedge Q)'$	$P' \wedge Q'$ $P' \vee Q'$	De Morgan
$P \rightarrow Q$	$P' \vee Q$	Implication - imp
P	$(P')'$	Double negation - dn
$P \leftrightarrow Q$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$	Definition of equivalence

Derivation Rules (contd.)

Inference Rules

From	Can Derive	Name of Rule

Derivation Rules (contd.)

Inference Rules

From	Can Derive	Name of Rule
P, $P \rightarrow Q$	Q	*Modus Ponens* (mp)

Derivation Rules (contd.)

Inference Rules

From	Can Derive	Name of Rule
$P, P \rightarrow Q$	Q	*Modus Ponens* (mp)
$P \rightarrow Q, Q'$	P'	*Modus Tollens* (mt)

Derivation Rules (contd.)

Inference Rules

From	Can Derive	Name of Rule
P, $P \rightarrow Q$	Q	*Modus Ponens* (mp)
$P \rightarrow Q$, Q'	P'	*Modus Tollens* (mt)
P, Q	$P \wedge Q$	Conjunction

Derivation Rules (contd.)

Inference Rules

From	Can Derive	Name of Rule
$P, P \rightarrow Q$	Q	*Modus Ponens* (mp)
$P \rightarrow Q, Q'$	P'	*Modus Tollens* (mt)
P, Q	$P \wedge Q$	Conjunction
$P \wedge Q$	P, Q	Simplification

Derivation Rules (contd.)

Inference Rules

From	Can Derive	Name of Rule
$P, P \to Q$	Q	*Modus Ponens* (mp)
$P \to Q, Q'$	P'	*Modus Tollens* (mt)
P, Q	$P \wedge Q$	Conjunction
$P \wedge Q$	P, Q	Simplification
P	$P \vee Q$	Addition

A proof derivation

Example

Argue that

$$[A \wedge (B \to C) \wedge [(A \wedge B) \to (D \vee C')] \wedge B] \to D$$

is a valid argument.

Proof

A proof derivation

Example

Argue that

$$[A \wedge (B \rightarrow C) \wedge [(A \wedge B) \rightarrow (D \vee C')] \wedge B] \rightarrow D$$

is a valid argument.

Proof

(i) A hypothesis.

A proof derivation

Example

Argue that

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is a valid argument.

Proof

(i) A hypothesis.

(ii) B hypothesis.

A proof derivation

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Argue that

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is a valid argument.

Proof

(i)	A	hypothesis.
(ii)	B	hypothesis.
(iii)	$B \rightarrow C$	hypothesis.

A proof derivation

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Argue that

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is a valid argument.

Proof

(i)	A	hypothesis.
(ii)	B	hypothesis.
(iii)	$B \to C$	hypothesis.
(iv)	C	(ii), (iii), Modus Ponens.

A proof derivation

Example

Argue that

$$[A \wedge (B \to C) \wedge [(A \wedge B) \to (D \vee C')] \wedge B] \to D$$

is a valid argument.

Proof

(i)	A	hypothesis.
(ii)	B	hypothesis.
(iii)	$B \to C$	hypothesis.
(iv)	C	(ii), (iii), Modus Ponens.
(v)	$A \wedge B$	(i), (ii), Conjunction.

A proof derivation

Example

Argue that

$$[A \wedge (B \rightarrow C) \wedge [(A \wedge B) \rightarrow (D \vee C')] \wedge B] \rightarrow D$$

is a valid argument.

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(i)	A	hypothesis.
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Two more rules

Deduction Method

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The argument $P_1 \wedge P_2 \ldots P_n \rightarrow (R \rightarrow S)$ is tautologically equivalent to the argument $P_1 \wedge P_2 \ldots P_n \wedge R \rightarrow S$.

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Using the Deduction Method, the above argument can be rewritten as:
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Now apply Modus Ponens twice!

Note

The rule $[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$ *is called hypothesis syllogism and can be used directly.*

Proving validity of Verbal Arguments

Methodology

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From I and $(I \rightarrow H)$, we can derive H using Modus Ponens.

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Let I denote the event that interest rates will drop. Let H denote the event that the housing market will improve. Let F denote the event that the federal discount rate will drop. The symbolic argument is

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From I and $(I \rightarrow H)$, we can derive H using Modus Ponens. From H and $(F \vee H')$ $(H \rightarrow F)$, we can derive F using Modus Ponens! □

One More Example

Example

Show that the argument $A' \wedge (B \rightarrow A) \rightarrow B'$ is valid.

Proof

Consider the following proof sequence:

One More Example

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Show that the argument $A' \wedge (B \to A) \to B'$ is valid.

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Consider the following proof sequence:

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(ii) $(B' \vee A)$ (i), implication.

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(v) A' hypothesis.

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Consider the following proof sequence:

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(v) A' hypothesis.

(vi) B' (iv), (v) Modus Ponens.

Outline

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Example

Prove that $F \ : \ (P \wedge Q) \rightarrow (P \vee \neg Q)$ is valid.

Semantic Argument (Example)

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Assume that F is not valid. Therefore, there exists an interpretation I, such that $I \not\models F$. Accordingly,

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(vii)	$I \models \perp$	(iv) and (v) are contradictory.

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Equivalence

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