Propositional Logic - Decision Procedures

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Outline

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Simple Decision Procedures

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Naive methods

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(i) Truth tables.

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(ii) Semantic arguments (proof tactics).

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The resultant clause is called the **resolvent** and it is deduced from the original formula. Note that if $\bot$ is ever deduced, then the original formula **must** be unsatisfiable. If every possible resolution produces a clause that is either an original clause or a previously deduced clause, then the original formula must be satisfiable.

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$$\frac{\{l\} \qquad\qquad C_1[\neg l]}{C_1[\bot]}$$

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Is the following formula satisfiable?

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Note

Assume that a variable P appears only positively or only negatively. An easy optimization is to remove clauses in which this variable appears.

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(ii)

$$(\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R).$$