Propositional Logic - Substitutions and Normal Forms

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23 January

Outline

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1. Review
2. Substitution

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Main Issues

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1. Propositions and Connectives.

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2. Semantics and interpretation.

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2. Semantics and interpretation.
3. Satisfiability and Validity.
4. Proof techniques.

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Note

All substitutions must be performed simultaneously.

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Let $F : P \wedge Q \rightarrow P \vee \neg Q$ and $\sigma : \{P \mapsto R,\ P \wedge Q \mapsto (P \rightarrow Q)\}$.

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where each $F_i \Leftrightarrow G_i$.

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Let $\sigma_1 : \{P \mapsto R, P \wedge Q \mapsto (P \rightarrow Q)\}$ and $\sigma_2 : \{P \mapsto S, S \mapsto Q\}$.

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Let $\sigma_1 : \{P \mapsto R, P \wedge Q \mapsto (P \rightarrow Q)\}$ and $\sigma_2 : \{P \mapsto S, S \mapsto Q\}$. Compute $\sigma_1\sigma_2$.

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(i) Negation Normal Form (NNF).
(ii) Disjunctive Normal Form (DNF).
(iii) Conjunctive Normal Form (CNF).

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Example

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Clearly, only six more clauses are created, as opposed to the 9 created by using the first method.

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Convert the following formula to CNF:

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