Discrete Mathematics 2 - Homework II

K. Subramani
LCSEE,
West Virginia University,
Morgantown, WV
{ksmani@csee.wvu.edu}

1 Instructions

1. The homework is due on March 7, in class.
2. Each question is worth 4 points, except question $4.$, which is worth 6 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.
4. The work must be entirely your own. You are expressly **prohibited** from consulting with colleagues or the internet (with the exception of the material on the course website).

2 Problems

1. Prove that the following argument is valid, using the inference rule method:

$$(\exists x)(\forall y)\ Q(x,y) \rightarrow (\forall y)(\exists x)\ Q(x,y).$$

2. Show the following argument is valid, using the semantic argument method:

$$(\exists x)[P(x) \rightarrow Q(x)] \rightarrow [(\forall x)\ P(x) \rightarrow (\exists x)\ Q(x)].$$

3. Consider the following formula in T_E (the theory of equality):

$$[f(g(x)) = g(f(x)) \wedge f(g(f(y))) = x \wedge f(y) = x] \rightarrow [g(f(x)) = x].$$

   Prove that the formula is valid in T_E or provide an interpretation in which it is falsified.

4. Consider a theory $T_{\mathbf{N}}$ with signature: $\Sigma_{\mathbf{N}} = \{0, \sigma, +, \times, \uparrow, =, <\}$, where,

   (a) 0 is a constant,

   (b) σ is a unary function,

   (c) $+$, $\times$ and $\uparrow$ are binary functions, and

   (d) $=$ and $<$ are binary predicates.

   The axiom set of $T_{\mathbf{N}}$ is the following:

   (a) $(\forall x)\ x = x$.

   (b) $(\forall x)(\forall y)\ (x = y) \rightarrow (y = x)$.

   (c) $(\forall x)(\forall y)(\forall z)\ (x = y) \wedge (y = z) \rightarrow (x = z)$.

(d) $(\forall \overline{x})(\forall \overline{y})\ (\wedge_{i=1}^{n}(x_i = y_i)) \rightarrow [f(\overline{x}) = f(\overline{y})]$.

(e) $(\forall \overline{x})(\forall \overline{y})\ (\wedge_{i=1}^{n}(x_i = y_i)) \rightarrow [p(\overline{x}) = p(\overline{y})]$.

(f) $(\forall x)\ (x < (x + \sigma(0)))$.

(g) $(\forall x)(\exists y)\ (x = y + y)$.

Note that $\overline{x} = (x_1, x_2, \ldots x_n)$, and $\overline{y} = (y_1, y_2, \ldots y_n)$.

Consider the following two interpretations for $T_{\mathbf{N}}$:

(a) I_N - In this interpretation, the domain is the set of all natural numbers, $0_N = 0$, σ_N is the successor function, $+_N$, $\times_N$ and $\uparrow_N$ stand for traditional addition, multiplication and exponentiation respectively, and $<_N$ stands for the usual "strictly less than".

(b) I_L - In this interpretation, the domain is the set $2^{\{0,1\}^*}$, i.e., the set of all possible *languages* over $\{0, 1\}$, $0_L = \emptyset$, $\sigma_L(l) = l^*$, $+_L$, $\times_L$ and $\uparrow_L$ stand for union, concatenation and intersection respectively, and $<_L$ stands for the set relation "is a subset of".

Do the interpretations I_N and I_L satisfy the axioms of $T_{\mathbf{N}}$? Justify your answer.

5. A **group** is a set $\mathcal{G}$, together with an operation $\circ$, that satisfies the following properties:

   (a) For each pair of elements $x, y \in \mathcal{G}$, the result of the operation $x \circ y$ is also in $\mathcal{G}$.

   (b) $\circ$ is associative.

   (c) There must exist an element $e \in \mathcal{G}$, such that for every element $x \in \mathcal{G}$, $x \circ e = e \circ x = x$.

   (d) For every element $x \in \mathcal{G}$, there exists an element $y \in \mathcal{G}$, such that $x \circ y = y \circ x = e$.

   Express the above axioms as a first-order theory.