# Multicut and Integer Multicommodity Flow in Trees

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# Outline

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   - Definitions
   - IP formulation
   - Algorithm

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**Definition (Minimum Multicut)**

We are given an undirected graph $G = (V, E)$ where each edge $e \in E$ has a non-negative capacity $c_e$ and a set of set of vertex pairs $S = \{(s_1, t_1), \ldots, (s_k, t_k)\}$.

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**Definition (Maximum Integer Multicommodity Flow)**

We are given a graph $G$ and set of vertex pairs $S$ as in the multicut problem. If each $(s_i, t_i)$ pair is assigned its own commodity then the goal is to maximize the total flow of the system when an integral amount each commodity is routed.

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This is precisely the fractional version of the multicommodity flow problem.

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## Relaxed Dual Conditions

For each $i \in \{1, \ldots, k\}$, we must have that $f_i \neq 0 \rightarrow \sum_{e \in p_i} x_e \leq 2$.

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**Definition (Depth (edge))**

Given two edges $e_1$ and $e_2$ along the path from a vertex $v$ to the root, $r$, the edge $e_1$ is *deeper* than $e_2$ if $e_1$ occurs before $e_2$ along the path.

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In the algorithm **f** represents the multicommodity flow and $D$ represents the edges in the multicut.

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3. Let $e_1, \ldots e_l$ be the order in which the edges were added to $D$.
4. For each $j = l, \ldots, 1$ if removing $e_j$ from $D$ preserves the cut remove it.

**Theorem**

*Let $(s_i, t_i)$ be a pair of verticies such that $f_i \neq 0$ and let $v = lca(s_i, t_i)$. We have that at most one edge on the path from $s_i$ to $v$ is in $D$ and at most one edge from $t_i$ to $v$ is in $D$.*

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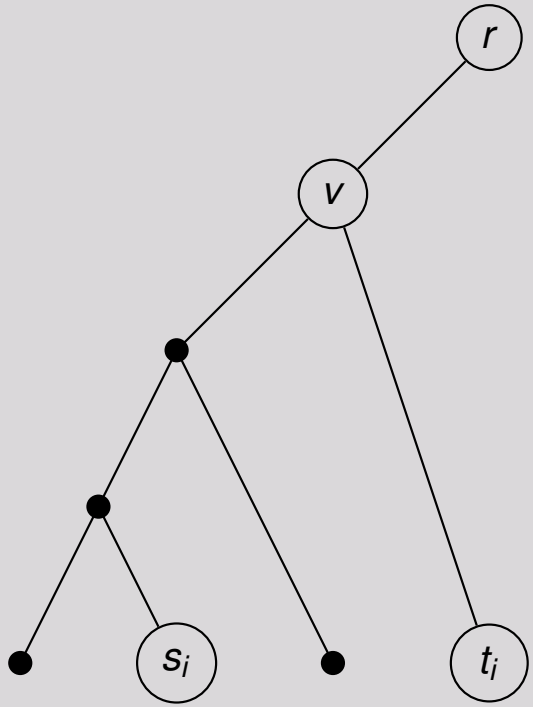
- This contradicts the fact that it was the only edge along the path from $s_j$ to $t_j$.

□

## Example



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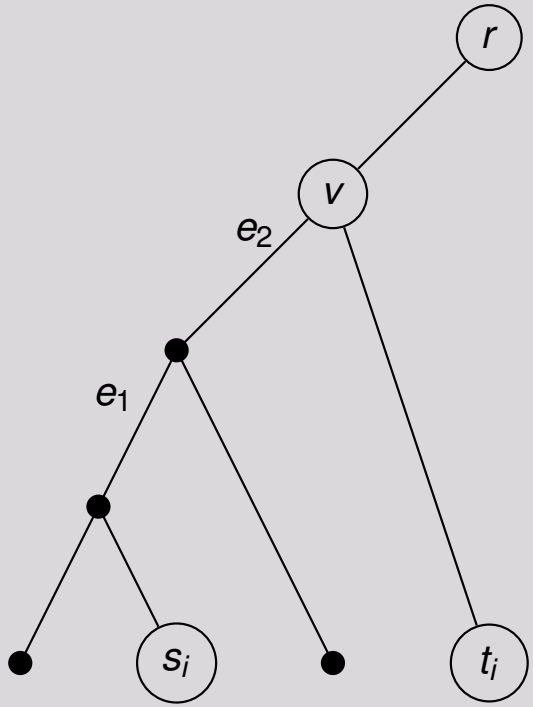

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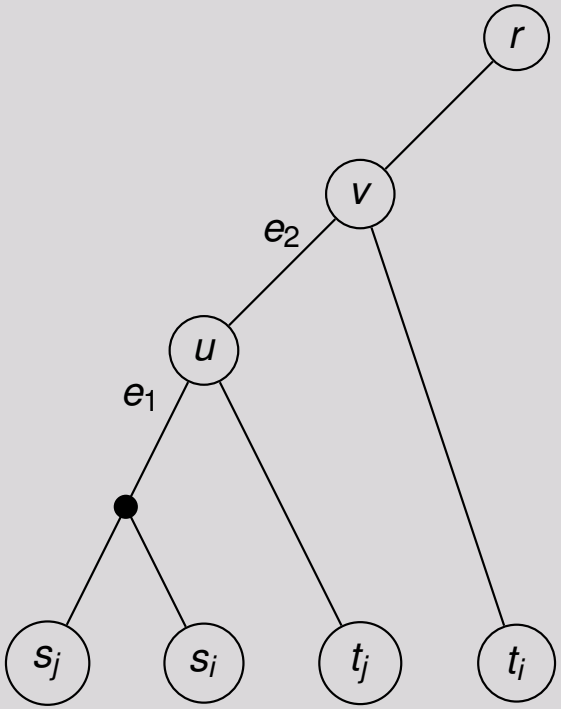

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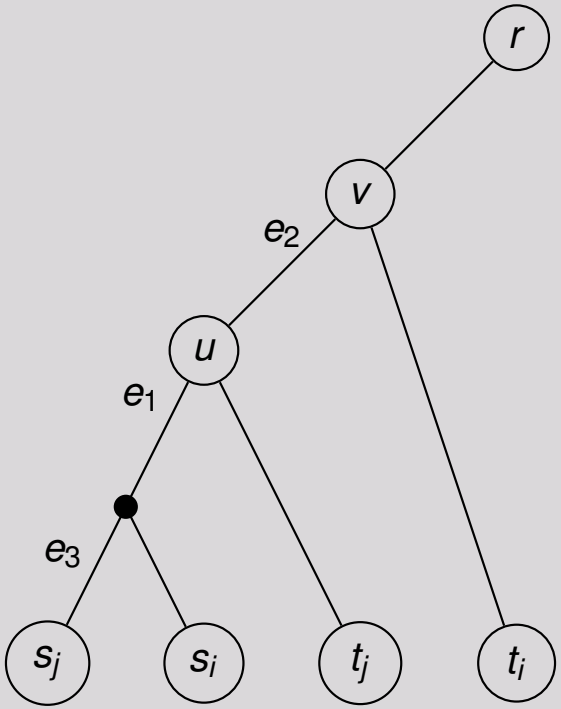

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- We remove edges from $D$ only if it still remains a valid multicut.
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- Each path with non-zero flow has at most two edges in $D$ so the relaxed dual slackness condition is satisfied.

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*The algorithm obtains an approximation factor of 2 for multicut and $\frac{1}{2}$ for integer multicommodity flows.*

**Proof.**

- The flow generated after all vericies have been processed is maximal.
- $D$ contains all saturated edges and so is a valid multicut.
- We remove edges from $D$ only if it still remains a valid multicut.
- Each edge in $D$ is saturated so the primal slackness condition is satisfied.
- Each path with non-zero flow has at most two edges in $D$ so the relaxed dual slackness condition is satisfied.
- The capacity of the multicut is at most twice the flow.

□