Scheduling on Unrelated Parallel Machines

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Outline

1. Scheduling on Unrelated Parallel Machines
   - Definition
   - IP formuation
   - Algorithm

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2. If job j_j is scheduled on the m_i machine the running time is p_{ij}.
3. We want to minimize the longest total running time over all machines.

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The main issue with this formulation is that it has an unbounded integrality gap. For example consider trying to schedule one job on m machines where $p_{i1} = m$ for all $i \in M$.

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- At most $n+m$ variables are nonzero.

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- Thus, $\beta \leq m$ and $\alpha \geq n - m$.

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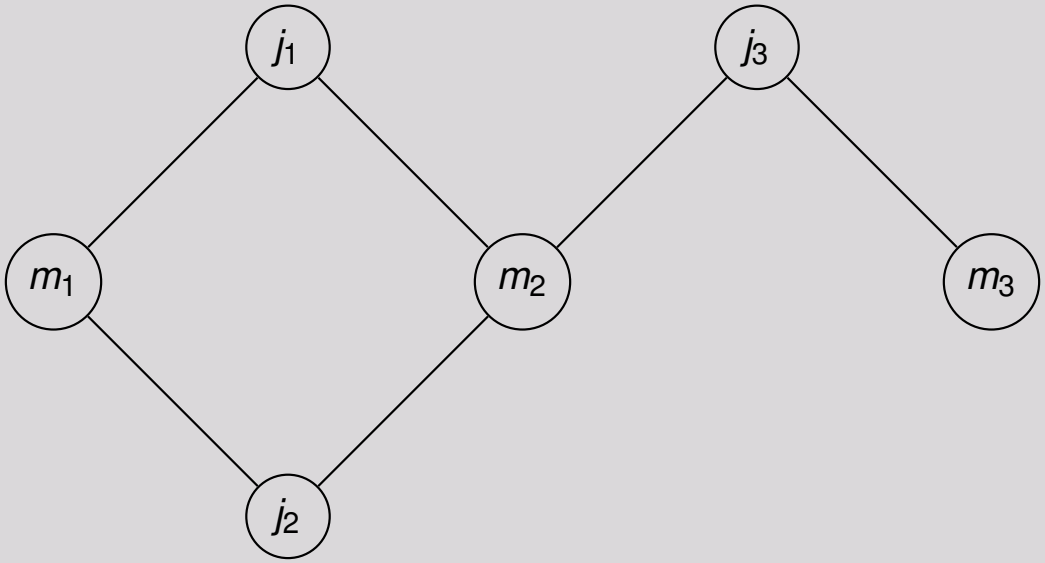

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- We have that the number of non-zero variables in $\mathbf{x_c}$ cannot exceed the number of machines plus the number of jobs in $LP_c(T)$.
- Thus the number of edges in G_c cannot exceed the number of verticies.

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- Alternating edges in these cycles form a matching in which each remaining job vertex is assigned to a unique remaining machine vertex.
- Thus at the end of this procedure each job in *F* has been uniquely assigned to one of the machines in *M*.

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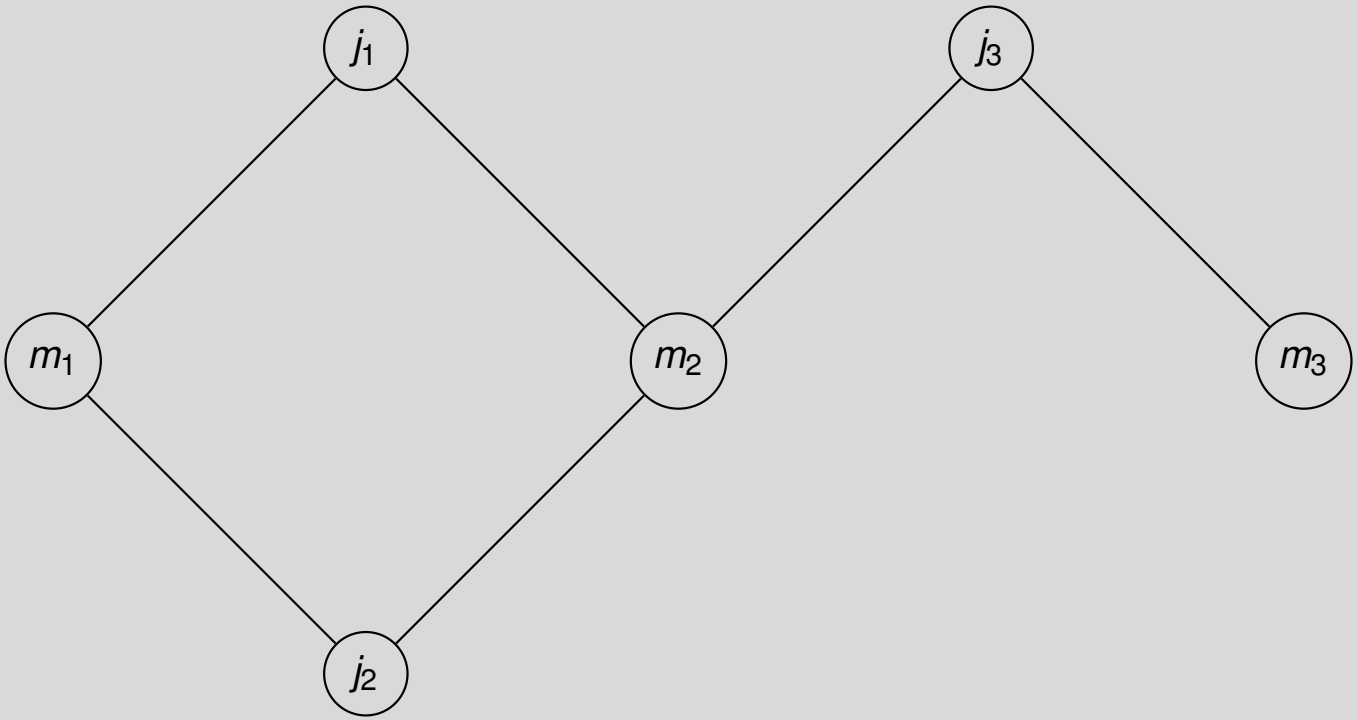

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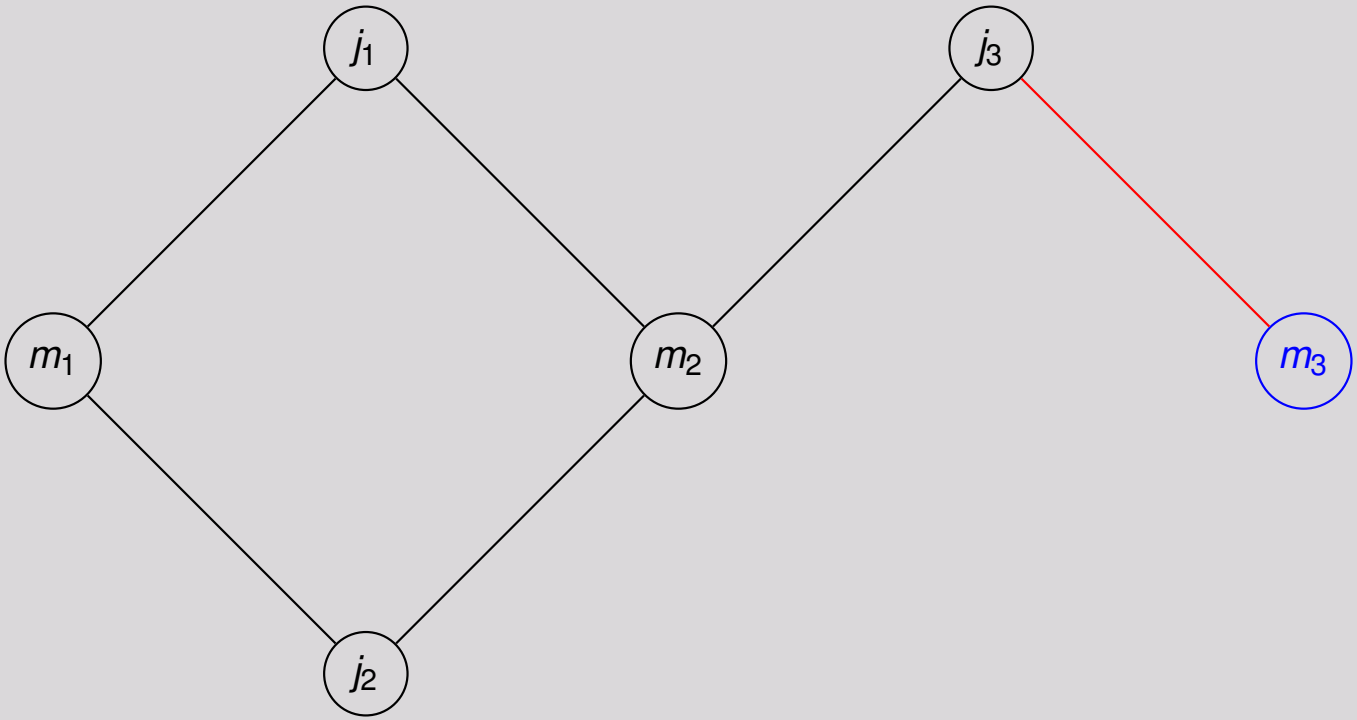

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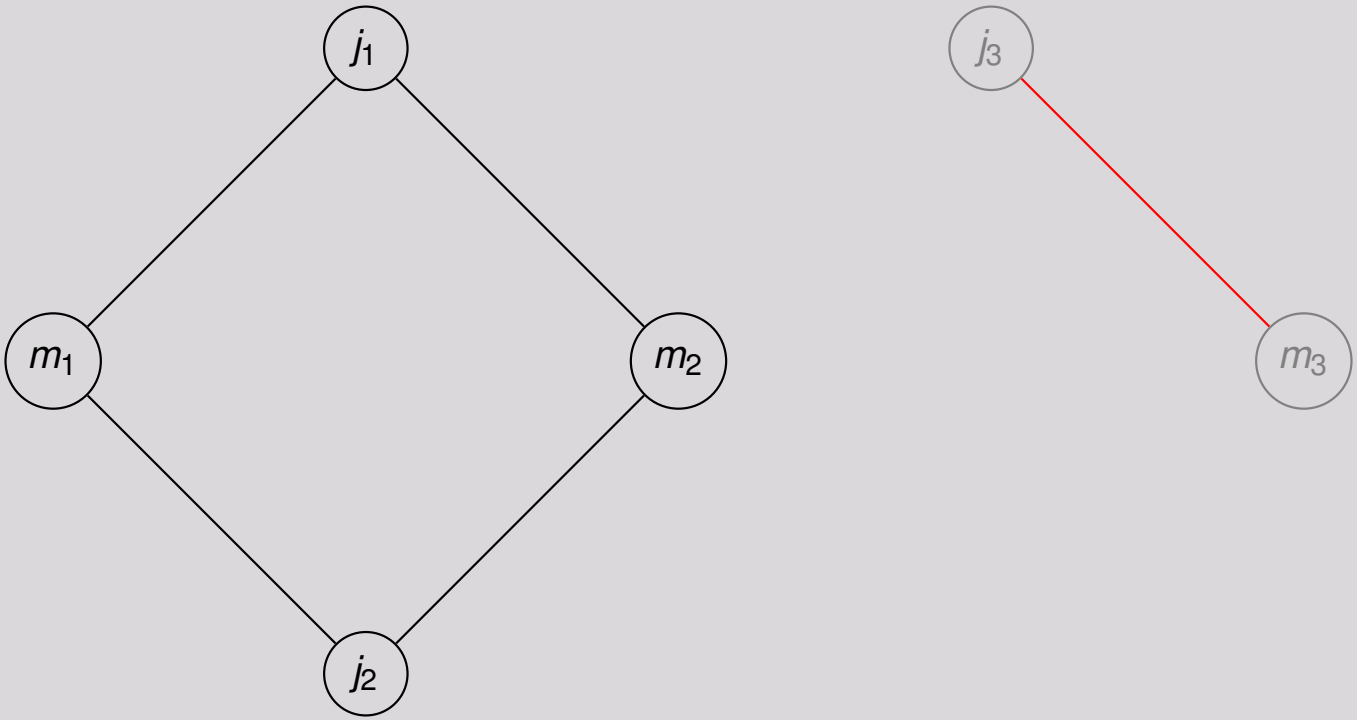

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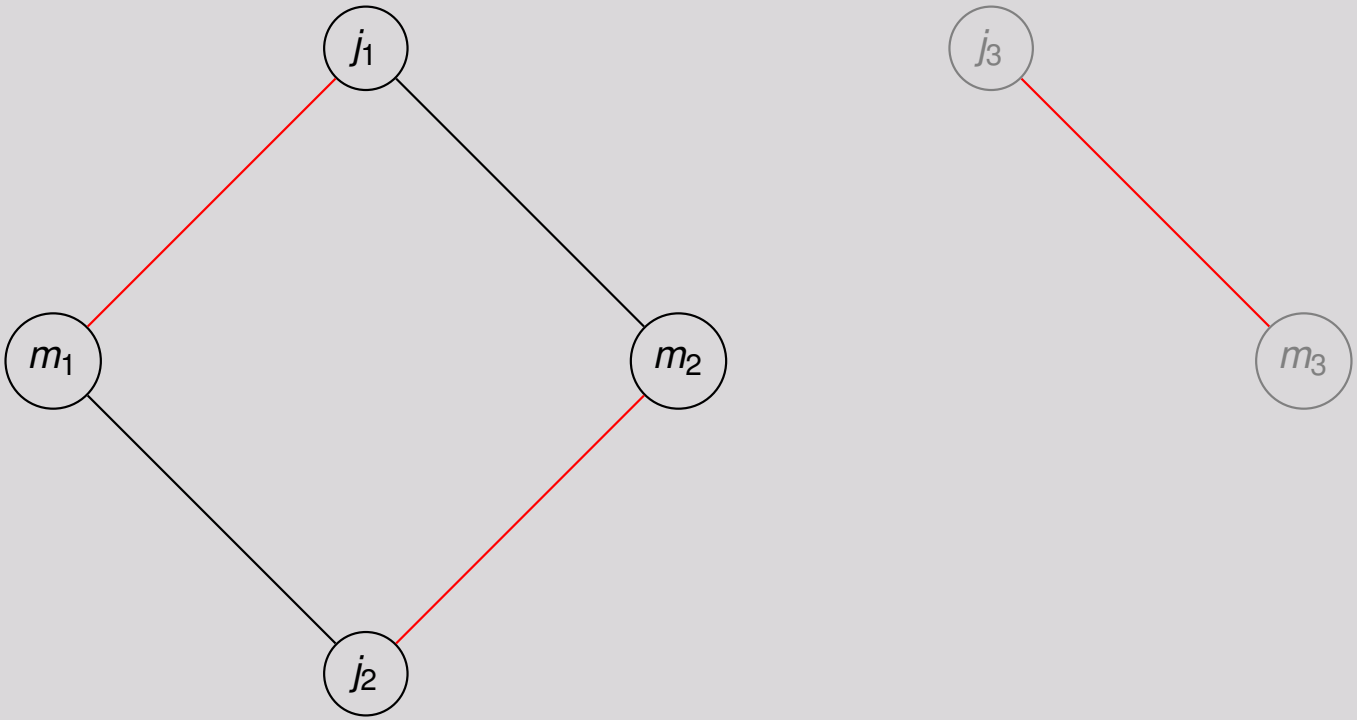

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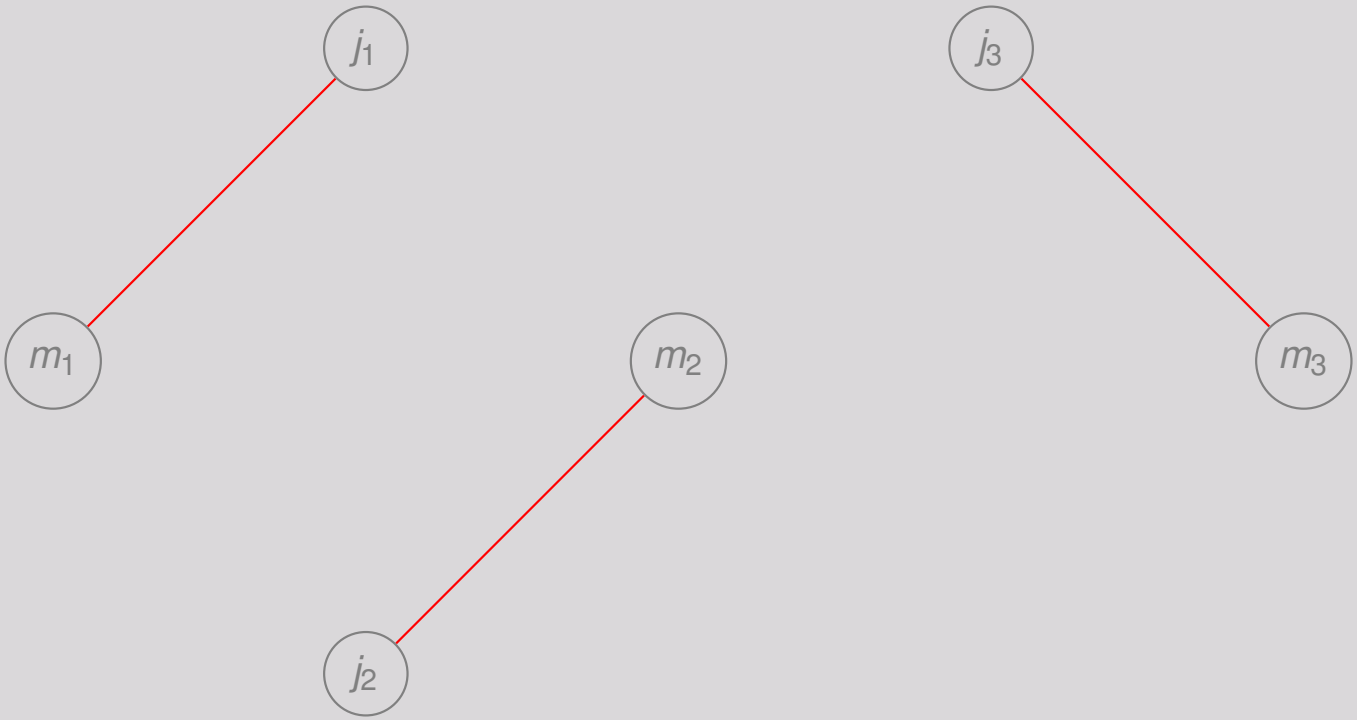

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5. Assign each fractionally set job to a unique machine according to the matching found in H.

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- Thus when restricted to integrally assigned jobs the makespan of the schedule is still at most T^*.
- According to the matching in H each machine is assigned at most one originally fractionally assigned job.
- This increases the makespan of the schedule by at most T^*.

Theorem

The algorithm has an approximation factor of 2.

Proof.

- We have that $T^* \leq OPT$.
- The makespan of the schedule corresponding to $\mathbf{x}$ is no more than T^*.
- Thus when restricted to integrally assigned jobs the makespan of the schedule is still at most T^*.
- According to the matching in H each machine is assigned at most one originally fractionally assigned job.
- This increases the makespan of the schedule by at most T^*.
- Thus the resultant schedule has total makespan at most $2 \cdot T^* \leq 2 \cdot OPT$.

□