The Facility Location Problem

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Outline

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Metric uncapacitated facility location problem

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3. The connection costs satisfy the triangle inequality.
4. The problem is to find a subset $I \subseteq F$ of facilities that should be opened, and a function $\phi : C \to I$ assigning cities to open facilities in such a way that the total cost of opening facilities and connecting cities to open facilities is minimized.

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3. Let (α, β) denote an optimal dual solution.

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3. Now consider condition (iv). If facility i is open, but $\phi(j) \neq i$, then $y_i \neq x_{ij}$. This means $\beta_{ij} = 0$.

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2. By condition (ii), each open facility must be fully paid for, i.e., if $i \in I$, then $\sum_{j:\phi(j)=i} \beta_{ij} = f_i$.
3. Now consider condition (iv). If facility i is open, but $\phi(j) \neq i$, then $y_i \neq x_{ij}$. This means $\beta_{ij} = 0$. In other words, city j does not contribute to opening any facility besides the one to which it is connected.

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4. All other primal conditions are maintained:
$\alpha_j - \beta_{\phi(j)j} = c_{\phi(j)j}$, for a directly connected city j,
and $\sum_{j:\phi(j)=i} \beta_{ij} = f_i$, for each open facility i.

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This phase consists of:

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2. Throughout this phase, the algorithm raises the dual variable α_j for each unconnected city j uniformly at unit rate, i.e., α_j will grow by 1 in unit time.

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3. When $\alpha_j = c_{ij}$ for some edge (i, j), the algorithm will declare this edge to be *tight*.

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4. Henceforth, dual variable β_{ij} will be raised uniformly, thus ensuring that the first constraint in the LP is not violated. β_{ij} goes towards paying for facility i. Each edge (i,j) such that $\beta_{ij} > 0$ is declared *special*.
5. Facility i is said to be *paid for* if $\sum_j \beta_{ij} = f_i$. If so, the algorithm declares this facility *temporarily open*.

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4. Henceforth, dual variable β_{ij} will be raised uniformly, thus ensuring that the first constraint in the LP is not violated. β_{ij} goes towards paying for facility i. Each edge (i,j) such that $\beta_{ij} > 0$ is declared *special*.
5. Facility i is said to be *paid for* if $\sum_j \beta_{ij} = f_i$. If so, the algorithm declares this facility *temporarily open*.
   1. All unconnected cities having tight edges to this facility are declared *connected* and facility i is declared the *connecting witness* for each of these cities. (*Notice that dual variables α_j of these cities are no longer raised*.)

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Phase 1

1. A notion of *time* is defined in this phase, so that each event can be associated with the time at which it happened; the phase starts at time 0.
    1. Initially, each city is defined to be *unconnected*.
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    2. In the future, as soon as an unconnected city j gets a tight edge to i, j will also be declared connected and i will be declared the connecting witness for j. (*Notice that $\beta_{ij} = 0$ and thus edge (i, j) is not special.*)

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    2. In the future, as soon as an unconnected city j gets a tight edge to i, j will also be declared connected and i will be declared the connecting witness for j. (*Notice that $\beta_{ij} = 0$ and thus edge (i, j) is not special.*)
6. When all cities are connected, this phase of the algorithm terminates. If several events happen simultaneously, the algorithm executes them in arbitrary order.

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    4. If $i' \in I$, again set $\phi(j) = i'$ and declare city j *directly connected*. (*Notice that in this case* $\beta_{i'j} = 0$).

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    4. If $i' \in I$, again set $\phi(j) = i'$ and declare city j *directly connected*. (*Notice that in this case* $\beta_{i'j} = 0$).
    5. In the remaining case that $i' \notin I$, let i be any neighbor of i' in graph H such that $i \in I$.

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   1. Let T^2 denote the graph that has edge (u, v) iff there is a path of length at most 2 between u and v in T, and let H be the subgraph of T^2 induced on F_t.
   2. Find any maximal independent set in H, say I.
   3. All facilities in the set I are declared *open*.
2. For city j, define $\mathscr{F}_j = \{i \in F_t \mid (i, j) \text{ is special}\}$.
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   3. Otherwise, consider tight edge (i', j) such that i' was the connecting witness for j.
   4. If $i' \in I$, again set $\phi(j) = i'$ and declare city j *directly connected*. (*Notice that in this case* $\beta_{i'j} = 0$).
   5. In the remaining case that $i' \notin I$, let i be any neighbor of i' in graph H such that $i \in I$.
   6. Set $\phi(j) = i$ and declare city j *indirectly connected*.

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   5. In the remaining case that $i' \notin I$, let i be any neighbor of i' in graph H such that $i \in I$.
   6. Set $\phi(j) = i$ and declare city j *indirectly connected*.
3. I and ϕ define primal integral solution: $x_{ij} = 1$ iff $\phi(j) = i$ and $y_i = 1$ iff $i \in I$.

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    5. In the remaining case that $i' \notin I$, let i be any neighbor of i' in graph H such that $i \in I$.
    6. Set $\phi(j) = i$ and declare city j *indirectly connected*.
3. I and ϕ define primal integral solution: $x_{ij} = 1$ iff $\phi(j) = i$ and $y_i = 1$ iff $i \in I$.
4. The values of α_j and β_{ij} obtained at the end of Phase 1 form a dual feasible solution.

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3. If j is indirectly connected, then $\alpha_j^f = 0$ and $\alpha_j^e = \alpha_j$
4. If j is directly connected, then the following must hold:

   $$\alpha_j = c_{ij} + \beta_{ij},$$

   where $i = \phi(j)$. Now, let $\alpha_j^f = \beta_{ij}$ and $\alpha_j^e = c_{ij}$.

Analysis

Lemma 1

Let $i \in I$. Then, $\sum_{j:\,\phi(j)=i} \alpha_j^f = f_i$.

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Proof

1. Since i is temporarily open at the end of Phase 1, it is completely paid for, i.e.,

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4. The lemma follows. $\square$

Analysis

Corollary 2

$$\sum_{i \in I} f_i = \sum_{j \in C} \alpha_j^f.$$

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$$\sum_{i \in I} f_i = \sum_{j \in C} \alpha_j^f.$$

Recall that α_j^f was defined to be 0 for indirectly connected cities. Thus, only the directly connected cities pay for the cost of opening facilities.

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Lemma 3

For an indirectly connected city j, $c_{ij} \leq 3\alpha_j^e$, where $i = \phi(j)$.

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1. Let i' be the connecting witness for city j.
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Proof

1. Let i' be the connecting witness for city j.
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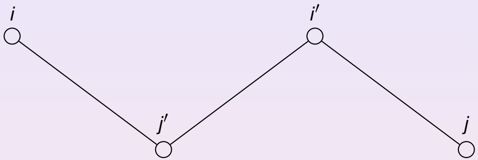
2. Since j is indirectly connected to i, (i, i') must be an edge in H.
3. In turn, there must be a city, say j', such that (i, j') and (i', j') are both special edges.
4. Let t_1 and t_2 be the times at which i and i' were declared temporarily open during Phase 1.

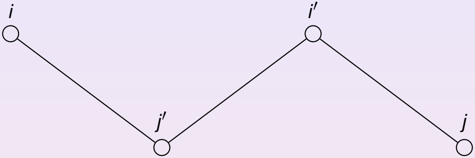
Analysis



Proof (*cont'd*)

1. Since edge (i', j) is tight, $\alpha_j \geq c_{i'j}$.

Analysis



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1. Since edge (i', j) is tight, $\alpha_j \geq c_{i'j}$.
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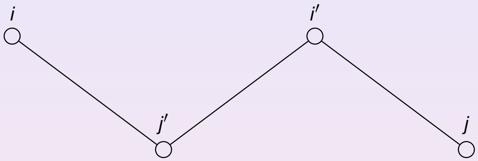

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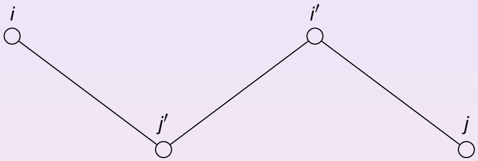
2. Since edges (i', j') and (i, j') are also tight, $\alpha_{j'} \geq c_{ij'}$ and $\alpha_{j'} \geq c_{i'j'}$.
3. Since both these edges are special, they must both have gone tight before either i or i' were declared temporarily open.

Analysis



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3. Since both these edges are special, they must both have gone tight before either i or i' were declared temporarily open.
4. Consider the time $\min(t_1, t_2)$.

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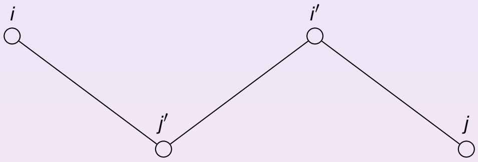

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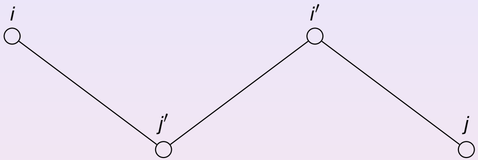
3. Since both these edges are special, they must both have gone tight before either i or i' were declared temporarily open.
4. Consider the time $\min(t_1, t_2)$. $\alpha_{j'}$ cannot be growing beyond this time.

Analysis



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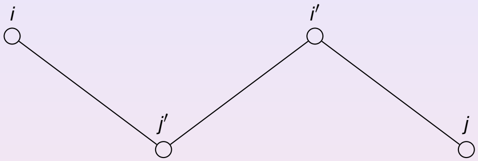
3. Since both these edges are special, they must both have gone tight before either i or i' were declared temporarily open.
4. Consider the time $\min(t_1, t_2)$. $\alpha_{j'}$ cannot be growing beyond this time. Therefore, $\alpha_{j'} \leq \min(t_1, t_2)$.

Analysis



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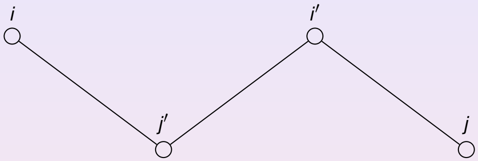
3. Since both these edges are special, they must both have gone tight before either i or i' were declared temporarily open.
4. Consider the time $\min(t_1, t_2)$. $\alpha_{j'}$ cannot be growing beyond this time. Therefore, $\alpha_{j'} \leq \min(t_1, t_2)$.
5. Finally, since i' is the connecting witness for j, $\alpha_j \geq t_2$.

Analysis



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4. Consider the time $\min(t_1, t_2)$. $\alpha_{j'}$ cannot be growing beyond this time. Therefore, $\alpha_{j'} \leq \min(t_1, t_2)$.
5. Finally, since i' is the connecting witness for j, $\alpha_j \geq t_2$.
6. Therefore, $\alpha_j \geq \alpha_{j'}$, and the required inequalities follow. $\square$

Analysis

Theorem 4

The primal and dual solutions constructed by the algorithm satisfy:

$$\sum_{i\in F, j\in C} c_{ij}\cdot x_{ij} + 3\sum_{i\in F} f_i \cdot y_i \leq 3\sum_{j\in C}\alpha_j .$$

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Theorem 4

The primal and dual solutions constructed by the algorithm satisfy:

$$\sum_{i \in F, j \in C} c_{ij} \cdot x_{ij} + 3 \sum_{i \in F} f_i \cdot y_i \leq 3 \sum_{j \in C} \alpha_j.$$

Proof

For a directly connected city j, $c_{ij} = \alpha_j^e \leq 3\alpha_j^e, \textit{where } \phi(j) = i$.

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The primal and dual solutions constructed by the algorithm satisfy:

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Proof

For a directly connected city j, $c_{ij} = \alpha_j^e \leq 3\alpha_j^e$, *where* $\phi(j) = i$.
Combined with Lemma 3, we get

$$\sum_{i\in F, j\in C} c_{ij}\cdot x_{ij} \leq 3\sum_{j\in C}\alpha_j^e.$$

Analysis

Theorem 4

The primal and dual solutions constructed by the algorithm satisfy:

$$\sum_{i \in F, j \in C} c_{ij} \cdot x_{ij} + 3 \sum_{i \in F} f_i \cdot y_i \leq 3 \sum_{j \in C} \alpha_j .$$

Proof

For a directly connected city j, $c_{ij} = \alpha_j^e \leq 3\alpha_j^e, where\ \phi(j) = i$.
Combined with Lemma 3, we get

$$\sum_{i \in F, j \in C} c_{ij} \cdot x_{ij} \leq 3 \sum_{j \in C} \alpha_j^e .$$

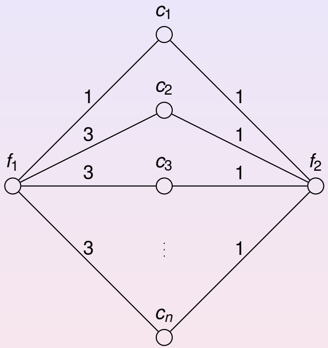
Adding this to the equality stated in Corollary 2, multiplied by 3, gives the theorem. $\square$

Analysis

Theorem 5

Algorithm 1 achieves an approximation factor of 3 for the facility location problem and has a running time of $O(m \log m)$.

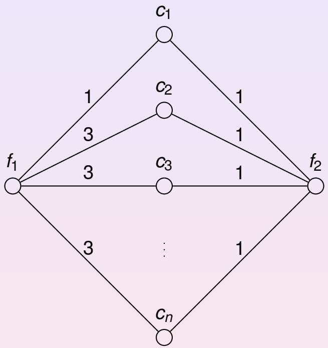
Tight example



Tight example

1. The graph has n cities, $c_1, c_2, \ldots, c_n$ and two facilities f_1 and f_2.

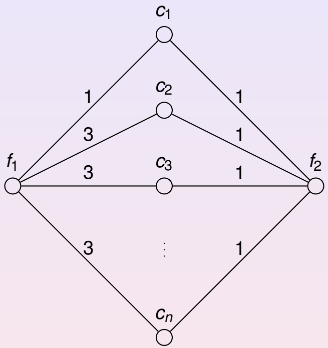
Tight example



Tight example

1. The graph has n cities, $c_1, c_2, \ldots, c_n$ and two facilities f_1 and f_2.
2. Each city is at a distance of 1 from f_2.

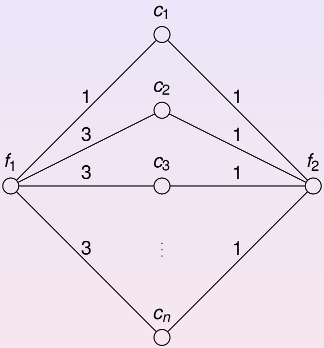
Tight example



Tight example

1. The graph has n cities, $c_1, c_2, \ldots, c_n$ and two facilities f_1 and f_2.
2. Each city is at a distance of 1 from f_2.
3. City c_1 is at a distance of 1 from f_1, and $c_2, \ldots, c_n$ are at a distance 3 from f_1.

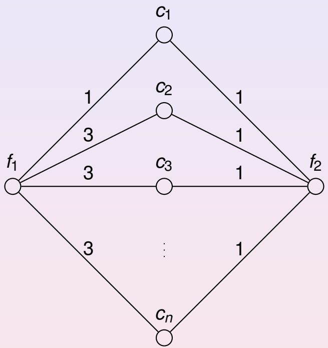
Tight example



Tight example

1. The graph has n cities, $c_1, c_2, \ldots, c_n$ and two facilities f_1 and f_2.
2. Each city is at a distance of 1 from f_2.
3. City c_1 is at a distance of 1 from f_1, and $c_2, \ldots, c_n$ are at a distance 3 from f_1.
4. The opening cost of f_1 and f_2 are ε and $(n+1)\varepsilon$, respectively, for a small number ε.

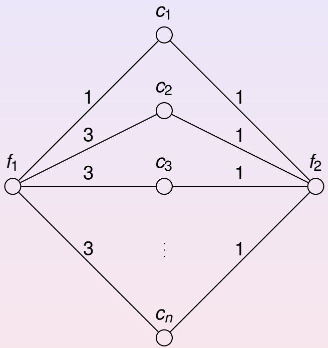
Tight example



Tight example (*cont'd*)

The optimal solution is to open f_2 and connect all cities to it, at a total cost of $(n+1)\varepsilon + n$.

Tight example



Tight example (*cont'd*)

The optimal solution is to open f_2 and connect all cities to it, at a total cost of $(n+1)\varepsilon + n$.

However, Algorithm 1 will open facility f_1 and connect all cities to it, at a total cost of $\varepsilon + 1 + 3(n-1)$.

References

Vazirani, V.V., 2003: *Approximation Algorithms*. Springer, 231 – 238.