Feedback Vertex Set Problem: Part 1

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A subset of V whose removal from G leaves an acyclic graph, is called a feedback set.

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Remark

Observe that any graph possesses a feedback set.

Recalling Some Topics

We need to recall the notion of a field, linear space over a field, and the directed sum of linear spaces.

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You may have a look at any book on General and Linear Algebra. Also wiki provides good source of information where one can recall basic concepts and facts.

The Field $GF(2)$ and the vector space $GF(2)^m$

Definition

Let $GF(2)$ denote the set $\{0,1\}$ with the operations $+$ and $\cdot$ defined on its elements by the following rules: $0+0=0$, $0+1=1$, $1+0=1$, $1+1=0$,

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For a positive integer $m \geq 1$, let $GF(2)^m$ denote the set of all vectors of length m, whose elements are from $GF(2)$. If $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$ are two vectors from $GF(2)^m$, then the vector $x + y$ is defined as follows:
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Remark

Observe that both of $L=\{0=(0,...,0)\}$ and $L=GF(2)^m$ form a linear space over $GF(2)$.

Linear Subspace, Linear independence and the dimension of a subspace

Definition

If $L, L' \subseteq GF(2)^m$ are two linear spaces over $GF(2)$, and $L \subseteq L'$, then L is said to be a subspace of L'.

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If $x_1, ..., x_k$ are k vectors from $GF(2)^m$, then these vectors are said to be linearly independent over $GF(2)$, if for any $\lambda_1, ..., \lambda_k \in GF(2)$ the equality $\lambda_1 \cdot x_1 + ... + \lambda_k \cdot x_k = 0 = (0, ..., 0)$ implies that $\lambda_1 = ... = \lambda_k = 0$.

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Definition

If $L \subseteq GF(2)^m$ is a linear space, then its dimension is defined to be the maximum number of linearly independent vectors from L.

Remark

Observe that the dimension of $L = \{0 = (0, \ldots, 0)\}$ *is zero, while it can be shown that the dimension of* $L = GF(2)^m$ *is m.*

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Definition

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Definition

The set of all $(= 2^k)$ linear combinations of $x_1, ..., x_k$ forms a linear space, which is called the span of the vectors $x_1, ..., x_k$. This is in fact the smallest subspace of $GF(2)^m$ that contains the vectors $x_1, ..., x_k$.

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Theorem

If $L = L' \oplus L''$, then the dimension of L is equal to the sum of dimensions L' and L''.

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The Cycle Space

The span of characteristic vectors corresponding to all simple cycles of G, is called cycle space of the graph G.

The Dimension of the Cycle Space

Theorem

The dimension of the cycle space of a graph $G = (V, E)$, denoted by $cyc(G)$, is given by the formula: $cyc(G) = |E| - |V| + comps(G)$, where $comps(G)$ is the number of connected components of G.

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Proof.

The proof can be found in chapter 6 of Vazirani's book. It uses the notion of an orthogonal subspace in Euclidean spaces. □