Feedback Vertex Set Problem: Part 2

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Cyclomatic Vertex Weights

Remark

Let $F = \{v_1, ..., v_f\}$ *be a feedback set, then* $cyc(G) = \sum_{i=1}^{f} \delta_{G_{i-1}}(v_i) \leq \sum_{v \in F} \delta_G(v)$*, where* $G = G_0$ *and* $G_i = G - \{v_1, ..., v_i\}$*.*

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A lower bound for OPT

If w is cyclomatic, and F is an optimal feedback set, then $c \cdot cyc(G) \leq c \cdot \sum_{v \in F} \delta_G(v) = w(F) = OPT$, hence $c \cdot cyc(G)$ is a lower bound for OPT.

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2-approximation algorithm for cyclomatic case

If w is cyclomatic, and F is a minimal feedback set, then $w(F) = c \cdot \sum_{v \in F} \delta_G(v) \leq 2 \cdot c \cdot cyc(G) \leq 2 \cdot OPT$.

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If w is cyclomatic, and F is a minimal feedback set, then $w(F) = c \cdot \sum_{v \in F} \delta_G(v) \leq 2 \cdot c \cdot cyc(G) \leq 2 \cdot OPT$. Hence a minimal feedback set approximates the optimal solution within a factor of 2 in case of cyclomatic weights.

Layering Applied to the General Case

Largest Cyclomatic and Residual Weight Functions

For a graph $G = (V, E)$ and a weight function w, let $c = \min_{v \in V}\{\frac{w(v)}{\delta_G(v)}\}$.

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$w'(v) = w(v) - t(v)$ is called the residual weight function.

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Decomposing into nested sequence of graphs and Cyclomatic Weight Functions

Repeated application of the above construction will result into a nested sequence $G_k \subset ... \subset G_1 \subset G_0 = G$ of induced subgraphs of G (where G_k is acyclic), and a largest cyclomatic weight function t_i defined on G_i ($i = 0, .., (k-1)$).

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Remark

Since the weight of a vertex v has been decomposed into the weights $t_0, t_1, ..., t_k$, we have $\sum_{i:v \in V(G_i)} t_i(v) = w(v)$.

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Let H be a subgraph of a graph $G = (V, E)$ induced on the vertex set $V' \subset V$. Let F be a minimal feedback vertex set in H, and let $F' \subseteq V - V'$ be a minimal set, such that $F \cup F'$ is a feedback vertex set for G.

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2-approximation algorithm for general weights

Formulation of the algorithm

- For the input graph $G = (V, E)$ and a weight function w defined on it, construct the nested sequence $G_k \subset ... \subset G_1 \subset G_0 = G$ of induced subgraphs of G (where G_k is acyclic), and a largest cyclomatic weight function t_i defined on G_i $(i = 1, .., k)$ defined above.

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- Since G_k is acyclic, $F_k = \emptyset$ is a minimal feedback vertex set.
- Repeatedly construct a minimal feedback vertex set in G_i, using the lemma.
- return $F = F_0$.

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$w(F_0) = \sum_{i=0}^{k} t_i(F_0 \cap V(G_i)) = \sum_{i=0}^{k} t_i(F_i) \leq 2 \cdot \sum_{i=0}^{k} OPT_i \leq 2 \cdot OPT$. □