Discrete Probability - Rudiments, Expectation and Variance

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Outline

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In the die tossing experiment, what is the probability of the event $\{2, 4, 6\}$?

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Some more examples

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A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy? Assume that the sample space is $S = \{(b,g),\ (b,b),\ (g,b),\ (g,g)\}$ and that all outcomes are equally likely.

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Two events E and F on a sample space S are said to be independent, if the occurrence of one does not affect the occurrence of the other.
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Outline

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Thus, the probability of an event E is the weighted average of the conditional probability of E, given that event F has occurred and the conditional probability of E, given that event F has not occurred, each conditional probability being given as much weight as the probability of the event that it is conditioned on, has of occurring.

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Therefore, $P(H \mid W) = \frac{\frac{1}{9}}{\frac{67}{198}} = \frac{22}{67}$, i.e., the conditional probability that the ball was drawn from Urn 1, given that it is white, is $\frac{22}{67}$.

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$$\begin{aligned} p(1) &= P\{X = 1\} = p \\ p(0) &= P\{X = 0\} = 1 - p \end{aligned}$$

where $0 \leq p \leq 1$ is the probability that the experiment results in a success.

The Binomial Random Variable

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Consider an experiment which consists of n independent Bernoulli trials, with the probability of success in each trial being p. If X is the random variable that counts the number of successes in the n trials, then X is said to be a Binomial Random Variable.

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$$p(i) = P\{X = i\} = C(n, i) \cdot p^i \cdot (1-p)^{n-i}, \ i = 0, 1, 2, \ldots n$$

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Consider the experiment of tossing four fair coins.

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Example (contd.)

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$$p(2) = C(4,2) \cdot (\frac{1}{2})^2$$

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$$p(2) \quad = \quad C(4,2) \cdot (\frac{1}{2})^2 \cdot (1 - \frac{1}{2})^2$$

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Solution

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$$\begin{aligned} p(2) &= C(4,2) \cdot (\frac{1}{2})^2 \cdot (1 - \frac{1}{2})^2 \\ &= \frac{3}{8} \end{aligned}$$

The Geometric Random Variable

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$$p(i) = P\{X = i\} = (1-p)^{i-1} \cdot p, \; i = 1, 2, \ldots$$

Outline

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Expectation of a Bernoulli Random Variable

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$$E[X] = 1 \cdot p + 0 \cdot (1 - p)$$

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\end{aligned}$$

□

Expectation of a Binomial Random Variable (contd.)

Example

Substituting $k = i - 1$, we get,

$$E[X] = n \cdot p \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot p^k \cdot (1-p)^{n-k-1}$$

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&= n \cdot p \cdot [p + (1-p)]^{n-1}, \text{ Binomial theorem}
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□

Expectation of a Geometric Random Variable (contd.)

Example

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$$E[X] = p \cdot \frac{d}{dq}[\sum_{i=1}^{\infty} q^i]$$

Expectation of a Geometric Random Variable (contd.)

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Solution:

$$\begin{aligned} E[X] &= p \cdot \frac{d}{dq}[\sum_{i=1}^{\infty} q^i] \\ &= p \cdot \frac{d}{dq}[\frac{q}{1-q}] \\ &= p \cdot \frac{(1-q) \cdot \frac{d}{dq}[q] - q \cdot \frac{d}{dq}[1-q]}{(1-q)^2} \end{aligned}$$

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□

Outline

Expectation of a function of a random variable

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Often times, we are interested in a function of the random variable, rather than the random variable itself.

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Example

Let X be a random variable, with the following pmf:

$$p(0) = 0.3, \ p(1) = 0.5, \ p(2) = 0.2$$

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Compute $E[X^2]$.

Expectation of functions of random variables (contd.)

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Solution

Let $Y = X^2$.

Expectation of functions of random variables (contd.)

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Expectation of functions of random variables (contd.)

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Expectation of functions of random variables (contd.)

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Expectation of functions of random variables (contd.)

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$$P\{Y = 0\} = P\{X^2 = 0\} = P\{X = 0\} = 0.3$$

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Expectation of functions of random variables (contd.)

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Accordingly,

$$E[Y] = E[X^2] = 0 \cdot 0.3 + 1 \cdot 0.5 + 4 \cdot 0.2 = 1.3$$

Expectation of functions - The Direct Approach

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Example

What is the expected value of the sum of the upturned faces, when two fair dice are tossed?

Another Application

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Compute the expected value of the Binomial random variable.

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Outline

Variance

Note

The **variance** *of a random variable measures the* **spread** *of its distribution.*

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$\mathbf{Var}[X] = E[X^2] - (E[X])^2$.

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Partial linearity of Variance

1. **Var**$(\sum_{i=1}^{n}(X_i)) = \sum_{i=1}^{n}$ **Var**(X_i), if $X_1, X_2, \ldots X_n$ are **independent** random variables.

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Observe that if X is a binomially distributed random variable with parameters n and p, then it can be expressed as a sum of n independent Bernoulli variables, i.e., $X = \sum_{i=1}^{n} X_i$, where each X_i is a Bernoulli random variable with parameter p.

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