Linear Programming Duality

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Outline

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Preliminaries

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Forms of linear programs, standard, canonical, etc. Feasible solution, optimal solution (z^*), unboundedness.

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6. *Consider $\mathbf{x} = (\frac{7}{4}, 0, \frac{11}{4})$ and $\mathbf{y} = (2, 1)$ for the example discussed above.*

Outline

Primal and Dual forms

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Forms

Primal (**P**):

$$z = \min \sum_{j=1}^{n} c_j \cdot x_j$$

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$$\sum_{j=1}^{n} c_j \cdot x_j \geq \sum_{i=1}^{m} b_i \cdot y_i$$

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Continuity of variables and the objective functions. □

Outline

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(vi) It follows that, $\mathbf{s}^*\mathbf{x}^* + \mathbf{y}^*\cdot\mathbf{t}^* = \mathbf{0}$.

(vii) Hence, $\mathbf{s}^*\cdot\mathbf{x}^* = \mathbf{0}$ and $\mathbf{y}^*\cdot\mathbf{t}^* = \mathbf{0}$,

Proof of Complementary Slackness

Proof.

(i) Recall that the primal is $\min_{\mathbf{A}\cdot\mathbf{x}\geq\mathbf{b},\mathbf{x}\geq\mathbf{0}} \mathbf{c}\cdot\mathbf{x}$ and the dual is $\max_{\mathbf{y}\cdot\mathbf{A}\leq\mathbf{c},\mathbf{y}\geq\mathbf{0}} \mathbf{b}\cdot\mathbf{y}$.

(ii) Let $(\mathbf{x}^*,\mathbf{y}^*)$ denote an optimal primal-dual pair.

(iii) Define $\mathbf{t}^* = \mathbf{A}\cdot\mathbf{x}^* - \mathbf{b}$ and $\mathbf{s}^* = \mathbf{c} - \mathbf{y}^*\cdot\mathbf{A}$.
Clearly, $\mathbf{t}^* \geq \mathbf{0}$ and $\mathbf{s}^* \geq \mathbf{0}$.

(iv) We have,

$$\begin{aligned}
\mathbf{c}\cdot\mathbf{x}^* &= (\mathbf{s}^* + \mathbf{y}^*\cdot\mathbf{A})\cdot\mathbf{x}^* \\
&= \mathbf{s}^*\mathbf{x}^* + \mathbf{y}^*\cdot\mathbf{A}\cdot\mathbf{x}^* \\
&= \mathbf{s}^*\mathbf{x}^* + \mathbf{y}^*\cdot(\mathbf{t}^* + \mathbf{b}) \\
&= \mathbf{s}^*\mathbf{x}^* + \mathbf{y}^*\cdot\mathbf{t}^* + \mathbf{y}^*\mathbf{b}
\end{aligned}$$

(v) But $\mathbf{c}\cdot\mathbf{x}^* = \mathbf{y}^*\cdot\mathbf{b}$.

(vi) It follows that, $\mathbf{s}^*\mathbf{x}^* + \mathbf{y}^*\cdot\mathbf{t}^* = \mathbf{0}$.

(vii) Hence, $\mathbf{s}^*\cdot\mathbf{x}^* = \mathbf{0}$ and $\mathbf{y}^*\cdot\mathbf{t}^* = \mathbf{0}$, since $\mathbf{x}^*,\mathbf{y}^*,\mathbf{s}^*,\mathbf{t}^* \geq \mathbf{0}$.

□

Interpretation of complementary slackness

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The Max-Flow Problem

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Problem statement

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find the maximum flow that can be sent from s to t, subject to:

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2. The total flow into a node is equal to the total flow out of the node, for all nodes other than s and t.

Preliminaries

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Maximize the flow on a new arc from t to s with capacity ∞.

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8. Consider an $s-t$ cut C. Every path from s to t contains at least one edge of C.

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Note

It may be possible to exploit the combinatorial structure of the dual program and design algorithms that are faster than general purpose linear programs.

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