Set-Cover approximation through Primal Dual Schema

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5. Cannot work directly for **NP-hard** problems, since the LP relaxations need not have integral optimal solutions.
6. However, a relaxation of the complementary slackness conditions helps in the derivation of approximation algorithms.

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Dual: $\forall j$, $1 \leq j \leq m$, either $y_j = 0$ or $t_j = 0$.

(v) We have,

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\mathbf{c}\cdot\mathbf{x}^* &= (\mathbf{s}^* + \mathbf{y}^*\cdot\mathbf{A})\cdot\mathbf{x}^* \\
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&= \mathbf{s}^*\mathbf{x}^* + \mathbf{y}^*\cdot(\mathbf{t}^* + \mathbf{b})
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Proof of Complementary Slackness

Proof.

(i) Recall that the primal is $\min_{\mathbf{A}\cdot\mathbf{x}\geq\mathbf{b},\mathbf{x}\geq\mathbf{0}} \mathbf{c}\cdot\mathbf{x}$ and the dual is $\max_{\mathbf{y}\cdot\mathbf{A}\leq\mathbf{c},\mathbf{y}\geq\mathbf{0}} \mathbf{b}\cdot\mathbf{y}$.

(ii) Let $(\mathbf{x}^*, \mathbf{y}^*)$ denote an optimal primal-dual pair.

(iii) Define $\mathbf{t}^* = \mathbf{A}\cdot\mathbf{x}^* - \mathbf{b}$ (surplus) and $\mathbf{s}^* = \mathbf{c} - \mathbf{y}^*\cdot\mathbf{A}$ (slack).
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4. Hence for $1 \leq i \leq n$, $x_i \cdot s_i = 0$, i.e., either $x_i = 0$ or $s_i = 0$.
5. Likewise, for $1 \leq j \leq n$, either $y_j = 0$ or $t_j = 0$.

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Let $\beta \geq 1$.

$$\forall i \; 1 \leq i \leq m: \;\text{ either } y_i = 0 \;\text{ or }\; b_i \leq \sum_{j=1}^{n} a_{ij} \cdot x_j \leq \beta \cdot b_i.$$

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7. The current primal solution is used to determine the improvement to the dual and vice versa.
8. Finally, the cost of the dual solution is used as a lower bound on OPT and the approximation guarantee of $\alpha \cdot \beta$ is obtained.

Preliminaries

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Note

For our scheme, we choose $\alpha = 1$ and $\beta = f$, where f is the frequency of the most frequent element.

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1. *A set S is tight under the current assignment to* $\mathbf{y}$*, if* $\sum_{e \in S} y_e = c(S)$.

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3. *Clearly no set can be overpacked, if dual feasibility is to be maintained.*

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Dual Conditions

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6. Output the set cover $\mathbf{x}$.

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4. By the Main Lemma, it follows that the approximation factor is f.

□

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9. This example achieves the bound of $f = n$.