Set-Cover approximation through LP-Rounding

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1 Preliminaries

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2. A Simple Rounding Algorithm

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3. A Randomized Rounding Algorithm
4. Half-integrality of Vertex Cover

Preliminaries

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If all weights are unity (or the same), the problem is called the Cardinality Set Cover problem.

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Lemma

The above algorithm achieves an approximation factor of f for the set cover problem.

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5. The rounding process increases x_S for each S by at most a factor of f.
6. Thus, the cost of C is at most f times the cost of the optimal fractional cover and hence at most f times the cost of the optimal integer cover!

□

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4. Output all sets S, such that $x_S = 1$.

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1. Run the randomized algorithm $c \cdot \ln n$ times independently and merge all the sets obtained into a set C', where $(\frac{1}{e})^{c \cdot \ln n} \leq \frac{1}{4 \cdot n}$.

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$$(\frac{1}{e})^{c \cdot \ln n} \leq \frac{1}{4 \cdot n}.$$

3. Summing over all elements, **Pr**[C' is not a valid cover] is at most $n \cdot \frac{1}{4 \cdot n}$

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4. Clearly, $\mathbf{E}[cost(C')] \leq OPT_f \cdot c \cdot \ln n$.

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4. Clearly, $\mathbf{E}[cost(C')] \leq OPT_f \cdot c \cdot \ln n$.
5. Applying Markov's inequality, $\mathbf{Pr}[cost(C') \geq 4 \cdot OPT_f \cdot c \cdot \ln n]$

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2. Observe that **Pr**[a is not covered by C'] is at most:
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3. Summing over all elements, **Pr**[C' is not a valid cover] is at most $n \cdot \frac{1}{4 \cdot n} = \frac{1}{4}$.
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$$y_v = \begin{cases} x_v + \varepsilon,\ x_v \in V_+ \\ x_v - \varepsilon,\ x_v \in V_- \\ x_v, \ \text{otherwise} \end{cases} \qquad z_v = \begin{cases} x_v - \varepsilon,\ x_v \in V_+ \\ x_v + \varepsilon,\ x_v \in V_- \\ x_v, \ \text{otherwise} \end{cases}$$

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