The Maximum Satisfiability problem

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MAX-SAT			
Outline			

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1 Preliminaries





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3 The randomized rounding algorithm
4 A $\frac{3}{4}$ factor algorithm

Satisfiability (SAT)

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Definition (Satisfiability (SAT))

Given a CNF formula $\phi = C_1 \land C_2 \ldots C_m$, over the variables $\{x_1, x_2, \ldots, x_n\}$, is there an assignment of $\{$ **true**, **false** $\}$ values to the literals, such that each clause C_i is satisfied?

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Example

$$\phi = (x_1, \bar{x_4}, \bar{x_7}) \\ (x_2, \bar{x_1}) \\ (x_3, \bar{x_1}, \bar{x_4})$$

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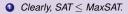
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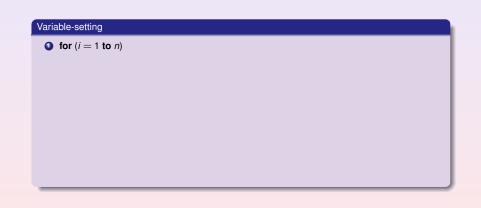
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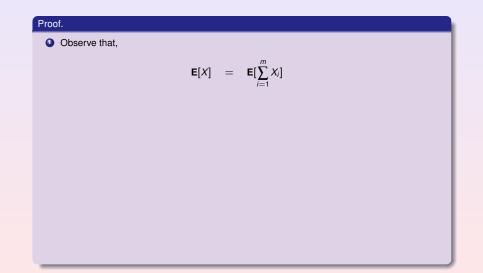
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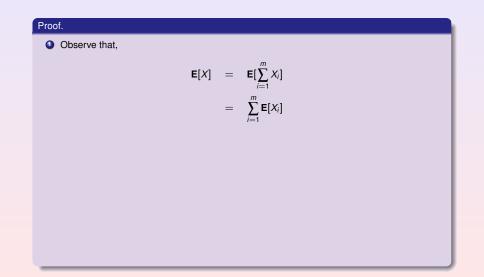
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- However, since X is a random variable, we focus on E[X].

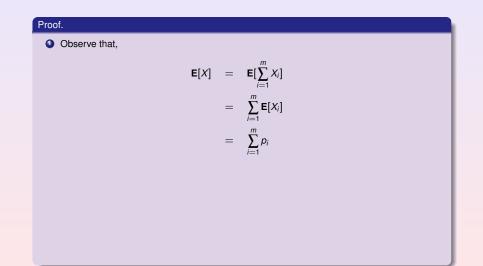
The variable setting algorithm











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- The probability that clause C_j remains unsatisfied after the rounding, is precisely the probability that each $y_i, i = 1, 2, ..., k$ was set to 0. This probability is clearly $\prod_{i=1}^{k} (1 \hat{y}_i)$.
- We thus need to show that

$$1-\prod_{i=1}^k (1-\hat{y}_i) \geq \beta_k \cdot \hat{z}_j.$$

Analysis (contd.)

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- Solution Note that $f(0) = 0 \ge \beta_k \cdot 0$ and $f(1) = \beta_k \ge \beta_k \cdot 1$.
- We apply the same logic to the linear function $g(z) = \beta_k \cdot z$ and the lemma follows.

Approximation bound

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Theorem

Let expected number of clauses satisfied by the randomized rounding algorithm is at least $(1 - \frac{1}{e}) \cdot OPT$.

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- Let X_i be 1, if clause C_i is satisfied, and 0 otherwise.
- **2** Let $X = \sum_{j=1}^{m} X_j$.
- **9** Let k_i be the number of literals in clause C_i and let p_i denote the probability that clause C_i is satisfied.

Final Steps (contd.)

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Proof.			

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L The randomized rounding algorithm

Final Steps (contd.)

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$$\mathbf{E}[X] \geq \sum_{i=1}^m (1-\frac{1}{e}) \cdot \hat{z}_i$$

$$\mathbf{E}[X] \geq \sum_{j=1}^{m} (1 - \frac{1}{e}) \cdot \hat{z}_j, \text{ since } \beta_k = 1 - (1 - \frac{1}{k})^k \geq (1 - \frac{1}{e}), \text{ for all positive integers } k$$

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```
3 return(max(n_1, n_2)).
```

Algorithm performance and clause width

$$k$$
 $\left(1-\frac{1}{2^k}\right)$ β_k

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$$\begin{array}{c|cc} k & (1-\frac{1}{2^k}) & \beta_k \\ \hline 1 & 0.5 & 1.0 \end{array}$$

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-	0 5	10
	0.5	1.0
2	0.75	0.75
~	0.75	0.70
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4	0.938	0.684

Algorithm	performance	and claus	se width
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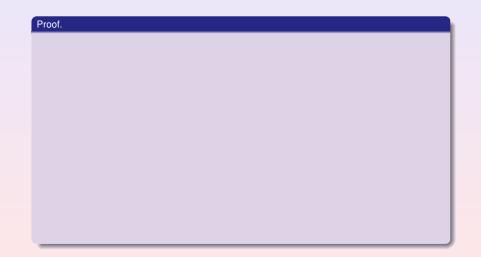
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Lemma

$$\max(n_1,n_2)\geq \frac{3}{4}\cdot OPT.$$



Proof.

• We will show that $\frac{n_1+n_2}{2} \ge \frac{3}{4} \cdot \sum_j \hat{z}_j$.

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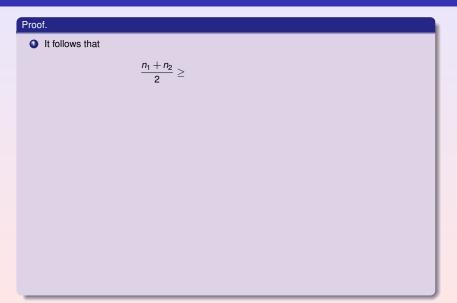
$$n_2 \geq \sum_k \sum_{C_j \in S^k} \beta_k \cdot \hat{z}_j.$$

Final Steps

Proof.

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