Tail Bounds

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Outline

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The tail bounds of a random variable X are concerned with the probability that it deviates signficantly from its expected value $E[X]$ on a run of the experiment.

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Consider the experiment of tossing a fair coin n times.

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The tail bounds of a random variable X are concerned with the probability that it deviates signficantly from its expected value $E[X]$ on a run of the experiment.

Example

Consider the experiment of tossing a fair coin n times. What is the probability that the number of heads exceeds $\frac{3}{4} \cdot n$?

Markov's inequality

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Let X be a non-negative random variable and let $c > 0$ be a positive constant. Then,

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Proof.

$$E[X] = \sum_x x \cdot P(X = x)$$

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Proof.

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\begin{aligned}
E[X] &= \sum_x x \cdot P(X = x) \\
&= \sum_{0 \leq x < c} x \cdot P(X = x) + \sum_{x \geq c} x \cdot P(X = x)
\end{aligned}
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&\geq \sum_{x \geq c} c \cdot P(X = x) \\
&= c \cdot P(X \geq c) \\
\Rightarrow P(X \geq c) &\leq \frac{E[X]}{c}
\end{aligned}$$

□

Markov's Inequality (contd.)

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Example (Application to coin tossing problem)

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Example (Application to coin tossing problem)

$$P(X \geq \frac{3n}{4}) \quad = \quad P(X \geq \frac{3}{2} \cdot \frac{n}{2})$$

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$$\begin{aligned} P(|X - E[X]| \geq a) &= P(|X - E[X]|^2 \geq a^2) \\ &\leq \frac{E[(X - E[X])^2]}{a^2}], \text{ Markov's inequality} \end{aligned}$$

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Chebyshev's theorem is alternatively stated as:

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Chebyshev's theorem is alternatively stated as:
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Chebyshev's inequality (contd.)

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\begin{aligned}
P(X \geq \frac{3n}{4}) &= P(X - \frac{n}{2} \geq \frac{n}{4}) \\
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\begin{aligned}
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&= P(|X - E[X]| \geq \frac{1}{2}E[X]) \\
&\leq \frac{\frac{n}{4}}{(\frac{1}{2})^2 \cdot (\frac{n}{2})^2} \\
&= \frac{4}{n}
\end{aligned}
$$

Chernoff Bounds

Theorem - Chernoff Bounds

Let $X_1, \ldots, X_n$ be a sequence of independent Poisson trials with $P(X_i) = p_i$, $X = \sum_{i=1}^{n} X_i$, and $\mu = E[X]$.

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1. for any $\delta > 0$,

$$P(X \geq (1+\delta)\mu) < (\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}})^{\mu}$$

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$$P(X \geq (1+\delta)\mu) < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$
2. for any $0 < \delta \leq 1$,
$$P(X \geq (1+\delta)\mu) \leq e^{-\frac{\mu\delta^2}{3}}$$
3. for $R \geq 6\mu$,
$$P(X \geq R) \leq 2^{-R}$$

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- The other two are easier to compute in many situations.

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Theorem - Chernoff Bounds - Deviation below the mean

Let $X_1, \ldots, X_n$ be a sequence of independent Poisson trials with $P(X_i) = p_i$, $X = \sum_{i=1}^{n} X_i$, and $\mu = E[X]$. Then, for $0 < \delta < 1$:

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1. $P(X \leq (1-\delta)\mu) \leq (\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}})^{\mu}$

Chernoff Bounds

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Note

- Again, the first bound is stronger.

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- Again, the first bound is stronger.
- The second is derived from the first.

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Note

- Again, the first bound is stronger.
- The second is derived from the first.
- The second is generally easier to use and sufficient.

Chernoff Bounds - Coin Flips

Example

Consider the probability of having no more than $\frac{n}{4}$ heads or no fewer than $\frac{3n}{4}$ tails in a sequence of n independent fair coin flips

Chernoff Bounds - Coin Flips

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Consider the probability of having no more than $\frac{n}{4}$ heads or no fewer than $\frac{3n}{4}$ tails in a sequence of n independent fair coin flips and let X be the number of heads.

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Chebyshev's bound:

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Chebyshev's bound:

$$P(|X - E[X]| \geq \alpha) \leq \frac{Var[X]}{\alpha^2}$$

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Recall that $Var[X] = \frac{n}{4}$.

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$$P(|X - \frac{n}{2}| \geq \frac{n}{4}) \quad \leq \quad \frac{Var[X]}{(\frac{n}{4})^2}$$

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$$= \frac{2}{e^{\frac{n}{24}}}$$

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Chernoff's inequality gives a bound that is exponentially smaller than the bound obtained using Chebyshev's inequality.

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Hence, whether we use Markov's, Chebyshev's or Chernoff bounds depends on the information we have available:

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Hence, whether we use Markov's, Chebyshev's or Chernoff bounds depends on the information we have available:

1. If we only know the expectation of X (i.e., $E[X]$), we use Markov's bound.
2. If we also know the variation of X (i.e., $Var[X]$), we use Chebyshev's bound.

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