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1 Summary of the paper

Definition 1.1 *A tree-decomposition of a graph $G = (X, C)$ is a pair (E, T), where $T = (I, F)$ is a tree, and $E = \{E_i : i \in I\}$ is a family of subsets of X such that*

- $\cup_{i \in I} E_i = X$,
- *for each edge $(x, y) \in C$, there is $i \in I$, such that $\{x, y\} \subseteq E_i$,*
- *for all $i, j, k \in I$ if k is on a unique $i - j$ path of T, then $E_i \cap E_j \subseteq E_k$.*

The width of the tree-decomposition (E, T) is equal to $\max_{i \in I} |E_i| - 1$. The tree-width $w(G)$ of a graph G is the minimum width over all tree-decompositions of G.

In the paper of Jegou and Terrioux the notion of Bag-Connected Tree-width of a graph is introduced.

Definition 1.2 *A bag-connected tree-decomposition of a graph $G = (X, C)$ is a pair (E, T), where $T = (I, F)$ is a tree, and $E = \{E_i : i \in I\}$ is a family of subsets of X such that*

- $\cup_{i \in I} E_i = X$,
- *for all $i \in I$ the subgraph of G induced by E_i is a connected graph,*
- *for each edge $(x, y) \in C$, there is $i \in I$, such that $\{x, y\} \subseteq E_i$,*
- *for all $i, j, k \in I$ if k is on a unique $i - j$ path of T, then $E_i \cap E_j \subseteq E_k$.*

The width of the tree-decomposition (E, T) is equal to $\max_{i \in I} |E_i| - 1$. The bag-connected tree-width $w_c(G)$ of a graph G is the minimum width over all bag-connected tree-decompositions of G.

It is known that the problem of calculation of tree-wdth of a graph is an **NP-hard** problem. In the paper the authors show that the same result can be obtained for bag-connected tree-width.

Theorem 1.1 *The problem of calculation of bag-connected tree-width is an* **NP-hard** *problem.*

Proof: The reduction is from the problem of calculation of tree-width. $\square$

The second result of the paper presents an algorithm that constructs some bag-connected tree-decomposition of a graph.

Theorem 1.2 *There exists an algorithm, which for any input graph $G = (V, E)$ with n vertices and m edges, constructs a bag-connected tree-decomposition in time $O(n \cdot (n + m))$.*

2 Our results

Using the reduction given in the paper, we were able to prove the following theorem.

Theorem 2.1 *If bag-connected tree-width can be approximated within a factor of C, then tree-width can be approximated within a factor of $2 \cdot C$.*

Proof: Let G be any graph. Consider a graph G' obtained from G by adding a vertex x which is adjacent to all vertices of G. As it is observed in the paper,

$$w_c(G') = w(G) + 1.$$

Now, let (E, T) be a bag-connected tree-decomposition of G', whose width is at most $C \cdot w_c(G')$. Observe that if we remove the vertex x from (E, T), we will get a tree-decomposition of G, whose width is at most the width of (E, T), which is at most

$$\leq C \cdot w_c(G') = C \cdot (w(G) + 1) \leq 2C \cdot w(G).$$

Hence the resulting tree-decomposition of G approximates the tree-width of G within a factor of $2 \cdot C$.

□

In http://arxiv.org/pdf/1109.4910v1.pdf Austrin, Pitassi and Wu have shown that Small Set Expansion Conjecture implies that tree-width cannot be approximated within a constant factor. Combined with the previous theorem, we get that bag-connected tree-width cannot have a constant-factor approximation under Small Set Expansion Conjecture.

It is known that

Theorem 2.2 *Let G be a connected graph. Then $w(G) \leq 1$ if and only if G is a tree.*

We were able to strengthen this result as follows:

Theorem 2.3 *Let G be a connected graph. Then $w_c(G) \leq 1$ if and only if G is a tree.*

Proof: Let G be a connected graph. Suppose that $w_c(G) \leq 1$. Since $w(G) \leq w_c(G) \leq 1$, we have that $w(G) \leq 1$, which combined with theorem 2.2 , implies that G is a tree.

Now assume that G is a tree. Let us show that G contains a bag-connected tree-decomposition of width one. For each edge e of G, let E_e be the set of end-vertices of e. Observe that since G is a tree, $\{E_e : e \in E(G)\}$ forms a bag-connected tree decomposition of G, whose width is one. □

It can be shown that

Theorem 2.4 *Let G be a connected graph. Then $w(G) \leq 2$ if and only if G is a K_4-free graph.*

Recall that a graph is defined to be K_4-free, if it does not contain a subgraph that is a minor (or subdivision) of the complete graph on four vertices.

The analogue of this theorem for bag-connected tree-width is wrong, that is the following theorem is wrong.

Theorem 2.5 *Let G be a connected graph. Then $w_c(G) \leq 2$ if and only if G is a K_4-free graph.*

In order to construct a counter-example, we will need the following proposition from Diestel's Graph Theory book.

Proposition: 2.1 *Let t_1t_2 be any edge of T and let T_1, T_2 be the components of $T - t_1t_2$, with $t_1 \in T_1$ and $t_2 \in T_2$. Then $E_{t_1} \cap E_{t_2}$ separates $U_1 = \cup_{t \in T_1} V_t$ from $U_2 = \cup_{t \in T_2} V_t$ in G.*

Now this proposition is applied as follows: we consider the graph C_n-the cycle of length n. Observe that it is a K_4-free graph, and $tw(C_n) = 2$. In order to disprove the theorem, it suffices to show that $tw_c(C_n)$ is not bounded by a constant.

Suppose it is, that is, $tw_c(C_n) = B$ for some constant B. Since the graphs $G[E_i]$ must be connected in the bag-connected tree-decomposition, we have that these graphs are just paths of length at most B. Now, if we take an edge t_1t_2 in T, we observe that the removal of $E_{t_1} \cap E_{t_2}$ results into a path, which means that U_1 and U_2 are not separated contradicting the statement of proposition.