Bag Connected Tree-Width - March 21, 2014

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1 Abstract

2 Introduction

A *constraint satisfaction problem* (CSP) is a triple (X, D, C), where $X = \{x_1, \cdots x_n\}$ is a set of n variables, $D = (D_{x_1}, \cdots, D_{x_n})$ is a list of finite domains of values, one per variable, and $C = \{C_1, \cdots, C_e\}$ is a finite set of e constraints. Each constraint C_i is a pair $(S(C_i), R(C_i))$, where $S(C_i) = \{x_{i_1}, \cdots, x_{i_l}\} \subseteq X$ is the scope of C_i, and $R(C_i) \subseteq D_{x_{i_1}} \times \cdots \times D_{x_{i_k}}$ is its *compatibility relation*. The *arity* of C_i is $|S(C_i)|$. A CSP is called *binary* if all constraints are of arity 2.

3 Statement of Problem

Definition 3.1 *A tree-decomposition of a graph $G = (X, C)$ is a pair (E, T), where $T = (I, F)$ is a tree, and $E = \{E_i : i \in I\}$ is a family of subsets of X such that*

- $\cup_{i \in I} E_i = X$,
- *for each edge $(x, y) \in C$, there is $i \in I$, such that $\{x, y\} \subseteq E_i$,*
- *for all $i, j, k \in I$ if k is on a unique $i - j$ path of T, then $E_i \cap E_j \subseteq E_k$.*

The width of the tree-decomposition (E, T) is equal to $\max_{i \in I} |E_i| - 1$. The tree-width $w(G)$ of a graph G is the minimum width over all tree-decompositions of G.

In the paper of Jegou and Terrioux the notion of Bag-Connected Tree-width of a graph is introduced.

Definition 3.2 *A bag-connected tree-decomposition of a graph $G = (X, C)$ is a pair (E, T), where $T = (I, F)$ is a tree, and $E = \{E_i : i \in I\}$ is a family of subsets of X such that*

- $\cup_{i \in I} E_i = X$,
- *for all $i \in I$ the subgraph of G induced by E_i is a connected graph,*
- *for each edge $(x, y) \in C$, there is $i \in I$, such that $\{x, y\} \subseteq E_i$,*
- *for all $i, j, k \in I$ if k is on a unique $i - j$ path of T, then $E_i \cap E_j \subseteq E_k$.*

The width of the tree-decomposition (E, T) is equal to $\max_{i \in I} |E_i| - 1$. The bag-connected tree-width $w_c(G)$ of a graph G is the minimum width over all bag-connected tree-decompositions of G.

Clearly, for any connected graph G the parameter $w_c(G)$ is well-defined (simply put all vertices into one cluster). Moreover, $w_c(G) \leq |V| - 1$.

The Statement of the Main Problem: For an input graph G, construct a bag-connected tree-decomposition of width $w_c(G)$.

4 Results

It is known that the problem of calculation of the tree-width of a graph is an **NP-hard** problem. In the paper the authors show that the same result can be obtained for bag-connected tree-width.

Theorem 4.1 *The problem of calculation of bag-connected tree-width of a graph is an* **NP-hard** *problem.*

Proof: The reduction is from the problem of calculation of tree-width. □

The second result of the paper presents an algorithm that constructs some bag-connected tree-decomposition of a graph.

Theorem 4.2 *There exists an algorithm, which for any input graph $G = (V, E)$ with n vertices and m edges, constructs a bag-connected tree-decomposition in time $O(n \cdot (n + m))$.*

5 Critiques