Approximation Algorithms - Homework I

K. Subramani
LCSEE,
West Virginia University,
Morgantown, WV
{ksmani@csee.wvu.edu}

1 Instructions

1. The homework is due on February 4, in class. (Extensions may be granted, depending upon requests).
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.
4. The work must be entirely your own, although you are encouraged to discuss your approaches with your colleagues and the instructor. You are expressly **prohibited** from consulting the internet.

2 Problems

1. **Linear Algebra:**
   Let $\mathbf{A}$ be a matrix in $\mathbb{R}^{m \times n}$ and let $\mathbf{b} \in \mathbb{R}^m$.
   (a) Prove that one and only one of the following statements can hold:
      i. $\exists \mathbf{x} \in \mathbb{R}^n \ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$.
      ii. $\exists \mathbf{z} \in \mathbb{R}^m_+ \ \mathbf{z} \cdot \mathbf{A} = \mathbf{0}, \text{ and } \mathbf{z} \cdot \mathbf{b} < 0$.
   (b) Prove that one and only of the following statements can hold:
      i. $\exists \mathbf{x} \in \mathbb{R}^n_+ \ \mathbf{A} \cdot \mathbf{x} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$.
      ii. $\exists \mathbf{y} \in \mathbb{R}^m \ \mathbf{y} \cdot \mathbf{A} > \mathbf{0}$.
2. **Approximation-factor Preserving Reductions:**
   (a) Recall the notion of approximation-factor preserving reductions discussed in class. Let Π_1 and Π_2 be two minimization problems such that there is an approximation-factor preserving reduction from Π_1 to Π_2. Show that if there is an α factor approximation for Π_2, then there is also an α factor approximation algorithm for Π_1.
   (b) Consider a randomized algorithm $\mathcal{A}$ for a minimization problem Π. On any input I, $\mathbf{E}[\mathcal{A}(I)] \leq \alpha \cdot OPT(I)$, for a fixed constant $\alpha > 1$. What is the best approximation guarantee that you can establish for problem Π, using algorithm $\mathcal{A}$?
   Hint: An approximation algorithm can provide a probabilistic guarantee. See page 346 of [Vaz02].
3. **Computational Complexity:**
   (a) Let L be some language. Argue that $L \in \mathbf{ZPP}$ if and only if $L \in (\mathbf{RP} \cap \mathbf{coRP})$.
   (b) Show that if $\mathbf{NP} \subseteq \mathbf{coRP}$, then $\mathbf{NP} \subseteq \mathbf{ZPP}$.

4. Consider the approach in Algorithm 2.1 for the Cardinality Vertex Cover problem. Will the approach always yield a feasible vertex cover? What is the quality of approximation of this algorithm?

Function CARDINALITY-VERTEX-COVER($G = \langle V, E \rangle$)
Let C denote the vertex cover.
$C = \emptyset$
while (there exists an edge in $G = \langle V, E \rangle$) **do**
  Pick the vertex v with the largest degree.
  Delete v from V and and add it to C.
  Delete all edges incident on v from E.
end while
Output C as the cover.

Algorithm 2.1: An algorithm for Cardinality Vertex Cover.

5. The MAXCUT problem is defined as follows: Given an undirected graph $G = \langle V, E \rangle$, partition the vertices in V into two sets A and B such that the number of inter-partition edges (edges going from A to B) is maximized. Algorithm 2.2 represents a strategy for the MAXCUT problem.

Function MAX-CUT($G = \langle V, E \rangle$)
Start with an arbitrary partition of V into sets A and B. In other words, every vertex is assigned to either A or B, in some arbitrary fashion.
while (there exists a flippable vertex v) **do**
  Flip v.
end while
Output $\langle A, B \rangle$ as the final partition.

Algorithm 2.2: An algorithm for the MAXCUT problem

Note that a vertex $v \in A$ (or B) is said to be *flippable*, if moving it to B (or A respectively), increases the cardinality of the inter-partition edges. Likewise, flipping a vertex, means moving it from the set to which it belongs to the other set. Argue that on any instance, Algorithm 2.2 terminates and produces a solution, whose value is at least half the optimal value.

References

[Vaz02] Vijay Vazirani. *Approximation Algorithms*. Springer Science Publishers, 1^{st} edition, 2002.