Approximation Algorithms - Homework II

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1 Instructions

1. The homework is due on March 11, in class.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.
4. The work must be entirely your own, although you are encouraged to discuss your approaches with your colleagues and the instructor. You are expressly **prohibited** from consulting the internet.

2 Problems

1. Let $G = \langle V, E \rangle$ denote a directed graph with nonnegative edge costs. The vertex set V is partitioned into two sets, viz., required and Steiner. One of the required vertices, r, is special. The problem is to find a minimum cost tree in G rooted into r, that contains all the required vertices and any subset of the Steiner vertices. Argue that this problem cannot have an approximation factor that is asymptotically better than $O(\log n)$, unless **P= NP**. (*Hint: Reduce Set covert to this problem in an approximation-preserving manner.*
2. Consider variants on the metric TSP problem in which the objective is to find a simple path containing all the vertices of the graph. Three different problems arise, depending on the number (0, 1, or 2) of endpoints of the path that are specified. Obtain the following approximation algorithms:
   (a) If zero or one endpoints are specified, obtain a 3/2 factor algorithm.
   (b) If both endpoints are specified, obtain a 5/3 factor algorithm.
3. Recall that in the multiway cut problem, we are given an undirected, positively weighted graph $G = \langle V, E, c \rangle$ and a set of terminals $\{s_1, s_2, \ldots s_k\}$. The goal is to find a minimum weight collection of edges that separates the terminal from one another. A natural greedy algorithm for computing a multiway cut is the following. Starting with G, compute minimum $s_i s_j$ cuts for all pairs s_i, s_j that are still connected and remove the lightest of these cuts. Repeat this until all pairs s_i, s_j are disconnected. Prove that this algorithm also achieves a guarantee of $(2 - \frac{2}{k})$.
4. Let $G = \langle V, E \rangle$ be a graph and let $\mathbf{w} : E \rightarrow R_+$ be an assignment of non-negative weights to the edges in E. For $u, v \in V$, let $f(u, v)$ denote the weight of a minimum $u - v$ cut in G. Prove the following:
   (a) If $f(u, v) \leq f(u, w) \leq f(v, w)$, then $f(u, v) = f(u, w)$, for all vertices $u, v, w \in V$.
   (b) There are at most $(n - 1)$ distinct min-cut values among the $\binom{n}{2}$ possible values over all vertex pairs, where $n = |V|$.
   (c) $\forall u, v, w \in V$, $f(u, v) \geq \min f(u, w), f(w, v)$.

(d) Prove that if the GomoryHu tree of G contains all $(n-1)$ distinct weights, then G can have only one minimum weight cut.

5. Let $G = \langle V, E \rangle$ be a complete undirected graph with edge costs satisfying the triangle inequality, and let k be a positive integer. The problem is to partition V into sets $V_1, V_2, \ldots V_k$ so as to minimize the costliest edge between two vertices in the same set, i.e., minimize

$$\max_{1 \leq i \leq k, u, v \in V_i} \mathrm{cost}(u, v).$$

(a) Give a factor 2 approximation algorithm for this problem, together with a tight example.

(b) Show that this problem cannot be approximated within a factor of $(2 - \epsilon)$, for any $\epsilon > 0$, unless **P = NP**.