Algorithmic Insights II - Greedy and Dynamic Programming

K. Subramani[1]

[1] Lane Department of Computer Science and Electrical Engineering
West Virginia University

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Outline

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Review

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Algorithmic Insights

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1. Recursion.

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2. Divide and Conquer.

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1. Recursion.
2. Divide and Conquer.
3. Greedy.

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5. Iterative approaches (Rewriting).

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Algorithmic Insights

1. Recursion.
2. Divide and Conquer.
3. Greedy.
4. Dynamic Programming.
5. Iterative approaches (Rewriting).
6. Transformations and reductions.

The Greedy Approach

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Main Idea

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5. Terminate when all items have been considered or a maximum feasible subset has been reached.

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5. Different orders of file storage give rise to different expected costs.
6. In what order should the files be stored, so that the expected cost is minimized?

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5. Thus, a non-ordered organization cannot be optimal.

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Algorithm 3.13: Kruskal's algorithm

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Let C denote a cut-set corresponding to some cut in an undirected graph G. There is an MST of G, which includes the lightest edge in C.

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4: **end for**

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5. Observe that α_k must be greater than α'_k.
6. Use an exchange argument.

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5: **if** $(S \cup \{J_i\}$ is feasible) **then**
6: $S \rightarrow S \cup \{J_i\}$.
7: **end if**

Scheduling with profits and deadlines

The problem

1. You are given n unit time jobs, J_i, $i = 1, 2, \ldots, n$.
2. Job J_i has a deadline d_i and a profit p_i.
3. If a job commences execution after its deadline, its profit is 0.

Schedule the jobs so as to maximize profit.

Greedy Algorithm

Order the jobs in descending order of profit.
Assume that $p_1 \geq p_2 \ldots \geq p_n$.
Let $S = \emptyset$.
for $(i = 1$ **t0** $n)$ **do**
    if $(S \cup \{J_i\}$ is feasible) **then**
        $S \rightarrow S \cup \{J_i\}$.
    end if
end for

Algorithm 3.28: Job scheduling

Correctness

Correctness

Theorem

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A set of jobs S is feasible if and only if the sequence obtained by ordering the jobs according to nondecreasing deadlines is feasible.

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Proof of correctness

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Proof of correctness

Exchange argument.

The process scheduling problem

The process scheduling problem

The Problem

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3. Process P_i must start at s_i and is guaranteed to finish at f_i.

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2. Associated with process P_i is its start time s_i and finish time f_i.
3. Process P_i must start at s_i and is guaranteed to finish at f_i.
4. Any machine can execute only one process at a time.

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5. Processes P_i and P_j are said to be non-conflicting if $f_i \leq s_j$ or $f_j \leq s_i$.

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6. Two processes cannot be scheduled on the same machine if they conflict.

The process scheduling problem

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5. Processes P_i and P_j are said to be non-conflicting if $f_i \leq s_j$ or $f_j \leq s_i$.
6. Two processes cannot be scheduled on the same machine if they conflict.

Schedule all the processes, while minimizing the number of machines used.

The Greedy Algorithm

The Greedy Algorithm

The Greedy Approach

The Greedy Algorithm

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1: Order the processes in non-decreasing order of start time.

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The Greedy Algorithm

The Greedy Approach

Order the processes in non-decreasing order of start time.
W.l.o.g. assume that $s_1 \leq s_2 \leq \ldots s_n$.
for $(i = 1$ **to** $n)$ **do**
    Assign P_i to the first available machine.

The Greedy Algorithm

The Greedy Approach

Order the processes in non-decreasing order of start time.
W.l.o.g. assume that $s_1 \leq s_2 \leq \ldots s_n$.
for $(i = 1$ **to** $n)$ **do**
    Assign P_i to the first available machine.
    if (no machine is available) **then**

The Greedy Algorithm

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2: W.l.o.g. assume that $s_1 \leq s_2 \leq \ldots s_n$.
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4:     Assign P_i to the first available machine.
5:     **if** (no machine is available) **then**
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Order the processes in non-decreasing order of start time.
W.l.o.g. assume that $s_1 \leq s_2 \leq \ldots s_n$.
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The Greedy Algorithm

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The Greedy Algorithm

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        Assign it to a new machine.
    end if
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Correctness

The Greedy Algorithm

The Greedy Approach

```
Order the processes in non-decreasing order of start time.
W.l.o.g. assume that s_1 ≤ s_2 ≤ ... s_n.
for (i = 1 to n) do
    Assign P_i to the first available machine.
    if (no machine is available) then
        Assign it to a new machine.
    end if
end for
```

Correctness

Assume that the greedy approach requires k machines, but that the optimal solution requires $(k-1)$ machines.

The Greedy Algorithm

The Greedy Approach

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Let process P_l be the first process assigned to machine k in the greedy approach.

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Correctness

Assume that the greedy approach requires k machines, but that the optimal solution requires $(k-1)$ machines.

Let process P_l be the first process assigned to machine k in the greedy approach.

Clearly, P_i conflicts with all the processes on the fist $(k-1)$ machines.

The Greedy Algorithm

The Greedy Approach

```
Order the processes in non-decreasing order of start time.
W.l.o.g. assume that s_1 ≤ s_2 ≤ ... s_n.
for (i = 1 to n) do
    Assign P_i to the first available machine.
    if (no machine is available) then
        Assign it to a new machine.
    end if
end for
```

Correctness

Assume that the greedy approach requires k machines, but that the optimal solution requires $(k-1)$ machines.

Let process P_l be the first process assigned to machine k in the greedy approach.

Clearly, P_i conflicts with all the processes on the fist $(k-1)$ machines.

But these processes also conflict with each other!

The minimum weight matroid problem

The minimum weight matroid problem

Definition

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A matroid M is a finite set $E(M)$ together with a subset $\mathcal{I}(M)$ of $2^{E(M)}$ that satisfies the following properties:

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2. $(Y \in \mathcal{I}(M)$

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2. $(Y \in \mathcal{I}(M) \wedge (X \subseteq Y) \Rightarrow X \in \mathcal{I}(M)$.

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The above axioms are called independence axioms.

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A maximal independent set is said to be a basis.

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2. Let w_i denote the weight of s_i.

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A maximal independent set is said to be a basis.

The Problem

1. Let $E(M) = \{s_1, s_2, \ldots, s_n\}$.
2. Let w_i denote the weight of s_i.

Find a basis of minimum weight.

The matroid lemma

The matroid lemma

Lemma

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Let S be a set where the family of independent sets forms a matroid.

The matroid lemma

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Let S be a set where the family of independent sets forms a matroid. Suppose an independent set F is contained in a minimum-weight basis.

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Let S be a set where the family of independent sets forms a matroid.
Suppose an independent set F is contained in a minimum-weight basis.
Let v be one of the lightest elements of S such that $F \cup \{v\}$ is also independent.

The matroid lemma

Lemma

Let S be a set where the family of independent sets forms a matroid.
Suppose an independent set F is contained in a minimum-weight basis.
Let v be one of the lightest elements of S such that $F \cup \{v\}$ is also independent.
Then $F \cup \{v\}$ is also contained in a minimum-weight basis.

Dynamic Programming

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Main ideas

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1. Characterize the structure of an optimal solution.

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Dynamic Programming

Main ideas

1. Characterize the structure of an optimal solution.
2. Recursively define the value of an optimal solution.
3. Compute the value of an optimal solution, typically in a bottom-up fashion.
4. Construct an optimal solution from computed information.

The Rod Cutting problem

The Rod Cutting problem

The Problem

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Given a rod of n inches, and a table of prices p_i, $i = 1, 2, \ldots, n$, determine the maximum revenue r_n obtainable by cutting up the rod and selling it into pieces.

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The Rod Cutting problem

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Example

The Rod Cutting problem

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Example

Length i	1	2	3	4	5	6	7
Price p_i	1	5	8	9	10	17	17

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Example

Length i	1	2	3	4	5	6	7
Price p_i	1	5	8	9	10	17	17

Compute r_i, $i = 1, 2, \ldots 6$.

Optimal substructure property

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Recurrence

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Observe that once the first cut is made, you get two independent subproblems which must be solved optimally.

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$$r_n = \max(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots r_{n-1} + r_1). \tag{1}$$

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Recurrence (1) can be expressed more succinctly as:

$$r_n \quad =$$

Optimal substructure property

Recurrence

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Unlike Divide-and-Conquer, the subproblems could overlap.

Recurrence (1) can be expressed more succinctly as:

$$\begin{aligned} r_n &= \max_{1 \le i \le n} (p_i + r_{n-i}) \\ r_0 &= 0 \end{aligned} \tag{2}$$

Optimal substructure property

Recurrence

Observe that once the first cut is made, you get two independent subproblems which must be solved optimally. (Why?)

This is called the optimal substructure property. Hence, we can write,

$$r_n = \max(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots r_{n-1} + r_1). \tag{1}$$

Unlike Divide-and-Conquer, the subproblems could overlap.

Recurrence (1) can be expressed more succinctly as:

$$\begin{aligned} r_n &= \max_{1 \le i \le n} (p_i + r_{n-i}) \\ r_0 &= 0 \end{aligned} \tag{2}$$

Why are Recurrence (1) and Recurrence (2) equivalent?

A recursive implementation

A recursive implementation

Recursive Algorithm

A recursive implementation

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A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)

A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)

1: **if** ($n = 0$) **then**

A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)

1: **if** ($n = 0$) **then**
2:     **return**(0).

A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)

```
if (n = 0) then
    return(0).
end if
```

A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)

1: **if** ($n = 0$) **then**
2: **return**(0).
3: **end if**
4: $q = -\infty$.

A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)

if $(n = 0)$ **then**
    return(0).
end if
$q = -\infty$.
for $(i = 1$ **to** $n)$ **do**

A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)

```
if (n = 0) then
    return(0).
end if
q = −∞.
for (i = 1 to n) do
    q = max(q,
```

A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)

if $(n = 0)$ **then**
    return(0).
end if
$q = -\infty$.
for $(i = 1$ **to** $n)$ **do**
    $q = \max(q, p[i] + \text{CUT-ROD}(p, n - i))$.

A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)

if ($n = 0$) **then**
    return(0).
end if
$q = -\infty$.
for ($i = 1$ **to** n) **do**
    $q = \max(q, p[i] +$ CUT-ROD$(p, n - i))$.
end for

Algorithm 4.12: The recursive rod-cutting algorithm

A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)

if ($n = 0$) **then**
  return(0).
end if
$q = -\infty$.
for ($i = 1$ **to** n) **do**
  $q = \max(q, p[i] + \text{CUT-ROD}(p, n - i))$.
end for

Algorithm 4.13: The recursive rod-cutting algorithm

Analysis

A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)
1: **if** $(n = 0)$ **then**
2: **return**(0).
3: **end if**
4: $q = -\infty$.
5: **for** $(i = 1$ **to** $n)$ **do**
6: $q = \max(q, p[i] + \text{CUT-ROD}(p, n - i))$.
7: **end for**

Algorithm 4.14: The recursive rod-cutting algorithm

Analysis

$$T(n) \quad =$$

A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)

```
if (n = 0) then
    return(0).
end if
q = −∞.
for (i = 1 to n) do
    q = max(q, p[i] + CUT-ROD(p, n − i)).
end for
```

Algorithm 4.15: The recursive rod-cutting algorithm

Analysis

$$T(n) = \begin{cases} 1, & \text{if } n = 0 \\ & \end{cases}$$

A recursive implementation

Recursive Algorithm

Function CUT-ROD(p, n)

1: **if** $(n = 0)$ **then**
2: **return**(0).
3: **end if**
4: $q = -\infty$.
5: **for** $(i = 1$ **to** $n)$ **do**
6: $q = \max(q, p[i] +$ CUT-ROD$(p, n - i))$.
7: **end for**

Algorithm 4.16: The recursive rod-cutting algorithm

Analysis

$$T(n) = \begin{cases} 1, & \text{if } n = 0 \\ 1 + \sum_{j=1}^{n} T(n-j), & \text{otherwise} \end{cases}$$

Analysis of the recursive algorithm

Analysis of the recursive algorithm

Analysis (contd.)

Analysis of the recursive algorithm

Analysis (contd.)

$$T(n) \quad =$$

Analysis of the recursive algorithm

Analysis (contd.)

$$T(n) = \begin{cases} 1, & \text{if } n = 0 \\ & \end{cases}$$

Analysis of the recursive algorithm

Analysis (contd.)

$$T(n) = \begin{cases} 1, & \text{if } n = 0 \\ 1 + \sum_{k=0}^{n-1} T(k), & \text{otherwise} \end{cases}$$

Analysis of the recursive algorithm

Analysis (contd.)

$$T(n) = \begin{cases} 1, & \text{if } n = 0 \\ 1 + \sum_{k=0}^{n-1} T(k), & \text{otherwise} \end{cases}$$

It is not hard to see that $T(n) =$

Analysis of the recursive algorithm

Analysis (contd.)

$$T(n) = \begin{cases} 1, & \text{if } n = 0 \\ 1 + \sum_{k=0}^{n-1} T(k), & \text{otherwise} \end{cases}$$

It is not hard to see that $T(n) = 2^n$.

The Bottom-up approach

The Bottom-up approach

The bottom-up algorithm

The Bottom-up approach

The bottom-up algorithm

Function BOTTOM-ROD-CUT(p, n)

The Bottom-up approach

The bottom-up algorithm

Function BOTTOM-ROD-CUT(p, n)

1: Let $r[0 \cdot \cdot n]$ be a new array.

The Bottom-up approach

The bottom-up algorithm

Function BOTTOM-ROD-CUT(p, n)

1: Let $r[0 \cdot \cdot n]$ be a new array.
2: $r[0] = 0$.

The Bottom-up approach

The bottom-up algorithm

Function BOTTOM-ROD-CUT(p, n)

1: Let $r[0 \cdot \cdot n]$ be a new array.
2: $r[0] = 0$.
3: **for** $(j = 1$ **to** $n)$ **do**

The Bottom-up approach

The bottom-up algorithm

Function BOTTOM-ROD-CUT(p, n)

```
Let r[0 · ·n] be a new array.
r[0] = 0.
for (j = 1 to n) do
    q = −∞.
```

The Bottom-up approach

The bottom-up algorithm

Function BOTTOM-ROD-CUT(p, n)

1: Let $r[0 \cdot \cdot n]$ be a new array.
2: $r[0] = 0$.
3: **for** ($j = 1$ **to** n) **do**
4: $q = -\infty$.
5: **for** ($i = 1$ **to** j) **do**

The Bottom-up approach

The bottom-up algorithm

Function BOTTOM-ROD-CUT(p, n)

1: Let $r[0 \cdot \cdot n]$ be a new array.
2: $r[0] = 0$.
3: **for** ($j = 1$ **to** n) **do**
4: $\quad q = -\infty$.
5: $\quad$ **for** ($i = 1$ **to** j) **do**
6: $\quad\quad q = \max(q, p[i] + r[j - i])$.

The Bottom-up approach

The bottom-up algorithm

Function BOTTOM-ROD-CUT(p, n)

```
Let r[0 · ·n] be a new array.
r[0] = 0.
for (j = 1 to n) do
    q = −∞.
    for (i = 1 to j) do
        q = max(q, p[i] + r[j − i]).
    end for
```

The Bottom-up approach

The bottom-up algorithm

Function BOTTOM-ROD-CUT(p, n)

```
Let r[0 · ·n] be a new array.
r[0] = 0.
for (j = 1 to n) do
    q = −∞.
    for (i = 1 to j) do
        q = max(q, p[i] + r[j − i]).
    end for
    r[j] = q.
```

The Bottom-up approach

The bottom-up algorithm

Function BOTTOM-ROD-CUT(p, n)

Let $r[0 \cdot \cdot n]$ be a new array.
$r[0] = 0$.
for ($j = 1$ **to** n) **do**
  $q = -\infty$.
  for ($i = 1$ **to** j) **do**
    $q = \max(q, p[i] + r[j - i])$.
  end for
  $r[j] = q$.
end for

The Bottom-up approach

The bottom-up algorithm

Function BOTTOM-ROD-CUT(p, n)

Let $r[0 \cdot \cdot n]$ be a new array.
$r[0] = 0$.
for ($j = 1$ **to** n) **do**
    $q = -\infty$.
    for ($i = 1$ **to** j) **do**
        $q = \max(q, p[i] + r[j - i])$.
    end for
    $r[j] = q$.
end for
return($r[n]$).

Algorithm 4.29: Bottom-up rod-cutting

Analyzing the bottom-up approach

Analyzing the bottom-up approach

Analysis

Analyzing the bottom-up approach

Analysis

The running time of the algorithm can be approximated by the number of times that Line (6) is executed.

Analyzing the bottom-up approach

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Accordingly,

$$T(n) =$$

Analyzing the bottom-up approach

Analysis

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Accordingly,

$$T(n) = \begin{cases} 0, & \text{if } n = 0 \\ & \end{cases}$$

Analyzing the bottom-up approach

Analysis

The running time of the algorithm can be approximated by the number of times that Line (6) is executed.

Accordingly,

$$T(n) = \begin{cases} 0, & \text{if } n = 0 \\ \sum_{j=1}^{n} \sum_{i=1}^{j} 1, & \text{otherwise} \end{cases}$$

Analyzing the bottom-up approach

Analysis

The running time of the algorithm can be approximated by the number of times that Line (6) is executed.

Accordingly,

$$T(n) = \begin{cases} 0, & \text{if } n = 0 \\ \sum_{j=1}^{n} \sum_{i=1}^{j} 1, & \text{otherwise} \end{cases}$$

It is not hard to see that $T(n) =$

Analyzing the bottom-up approach

Analysis

The running time of the algorithm can be approximated by the number of times that Line (6) is executed.

Accordingly,

$$T(n) = \begin{cases} 0, & \text{if } n = 0 \\ \sum_{j=1}^{n} \sum_{i=1}^{j} 1, & \text{otherwise} \end{cases}$$

It is not hard to see that $T(n) = \Theta(n^2)$.

Reconstructing the Solution

Reconstructing the Solution

The bottom-up algorithm with solution

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

1: Let $r[0 \cdot \cdot n]$ and $s[0 \cdot \cdot n]$ be new arrays.

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

1: Let $r[0 \cdot \cdot n]$ and $s[0 \cdot \cdot n]$ be new arrays.
2: $r[0] = 0$.

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

1: Let $r[0 \cdot \cdot n]$ and $s[0 \cdot \cdot n]$ be new arrays.
2: $r[0] = 0$.
3: **for** ($j = 1$ **to** n) **do**

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

1: Let $r[0 \cdot \cdot n]$ and $s[0 \cdot \cdot n]$ be new arrays.
2: $r[0] = 0$.
3: **for** ($j = 1$ **to** n) **do**
4: $\quad q = -\infty$.

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

1: Let $r[0 \cdot \cdot n]$ and $s[0 \cdot \cdot n]$ be new arrays.
2: $r[0] = 0$.
3: **for** ($j = 1$ **to** n) **do**
4: $\quad q = -\infty$.
5: $\quad$ **for** ($i = 1$ **to** j) **do**

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

Let $r[0 \cdot \cdot n]$ and $s[0 \cdot \cdot n]$ be new arrays.
$r[0] = 0$.
for ($j = 1$ **to** n) **do**
    $q = -\infty$.
    for ($i = 1$ **to** j) **do**
        if ($q < p[i] + r[j - i]$) **then**

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

```
Let r[0 · ·n] and s[0 · ·n] be new arrays.
r[0] = 0.
for (j = 1 to n) do
    q = −∞.
    for (i = 1 to j) do
        if (q < p[i] + r[j − i]) then
            q = p[i] + r[j − i].
```

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

Let $r[0 \cdot \cdot n]$ and $s[0 \cdot \cdot n]$ be new arrays.
$r[0] = 0$.
for $(j = 1$ **to** $n)$ **do**
    $q = -\infty$.
    for $(i = 1$ **to** $j)$ **do**
        if $(q < p[i] + r[j - i])$ **then**
            $q = p[i] + r[j - i]$.
            $s[j] = i$. {The unsplittable left side is recorded.}

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

Let $r[0 \cdot \cdot n]$ and $s[0 \cdot \cdot n]$ be new arrays.
$r[0] = 0$.
for ($j = 1$ **to** n) **do**
    $q = -\infty$.
    for ($i = 1$ **to** j) **do**
        if ($q < p[i] + r[j - i]$) **then**
            $q = p[i] + r[j - i]$.
            $s[j] = i$. {The unsplittable left side is recorded.}
        end if

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

```
Let r[0 · ·n] and s[0 · ·n] be new arrays.
r[0] = 0.
for (j = 1 to n) do
    q = −∞.
    for (i = 1 to j) do
        if (q < p[i] + r[j − i]) then
            q = p[i] + r[j − i].
            s[j] = i. {The unsplittable left side is recorded.}
        end if
    end for
```

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

```
Let r[0 · ·n] and s[0 · ·n] be new arrays.
r[0] = 0.
for (j = 1 to n) do
    q = −∞.
    for (i = 1 to j) do
        if (q < p[i] + r[j − i]) then
            q = p[i] + r[j − i].
            s[j] = i. {The unsplittable left side is recorded.}
        end if
    end for
    r[j] = q.
```

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

Let $r[0 \cdot \cdot n]$ and $s[0 \cdot \cdot n]$ be new arrays.
$r[0] = 0$.
for ($j = 1$ **to** n) **do**
    $q = -\infty$.
    for ($i = 1$ **to** j) **do**
        if ($q < p[i] + r[j - i]$) **then**
            $q = p[i] + r[j - i]$.
            $s[j] = i$. {The unsplittable left side is recorded.}
        end if
    end for
    $r[j] = q$.
end for

Reconstructing the Solution

The bottom-up algorithm with solution

Function BOTTOM-ROD-CUT(p, n)

```
Let r[0··n] and s[0··n] be new arrays.
r[0] = 0.
for (j = 1 to n) do
    q = −∞.
    for (i = 1 to j) do
        if (q < p[i] + r[j − i]) then
            q = p[i] + r[j − i].
            s[j] = i. {The unsplittable left side is recorded.}
        end if
    end for
    r[j] = q.
end for
return(r[n]).
```

Algorithm 4.45: Bottom-up rod-cutting

Outputting the solution

Outputting the solution

Printing the Solution

Outputting the solution

Printing the Solution

Function PRINT-SOLUTION(p, n)

Outputting the solution

Printing the Solution

Function PRINT-SOLUTION(p, n)

1: **while** ($n > 0$) **do**

Outputting the solution

Printing the Solution

Function PRINT-SOLUTION(p, n)

```
1: while (n > 0) do
2:     print s[n].
```

Outputting the solution

Printing the Solution

Function PRINT-SOLUTION(p, n)

while ($n > 0$) **do**
    print $s[n]$.
    $n = n - s[n]$.

Outputting the solution

Printing the Solution

Function PRINT-SOLUTION(p, n)

```
while (n > 0) do
    print s[n].
    n = n − s[n].
end while
```

Algorithm 4.52: Extracting the solution

The Matrix Chain Multiplication problem

The Matrix Chain Multiplication problem

The Problem

You are required to compute the matrix product $A_1 \cdot A_2 \cdots A_n$,

The Matrix Chain Multiplication problem

The Problem

You are required to compute the matrix product $A_1 \cdot A_2 \cdots A_n$, where matrix A_i has dimensions $d_{i-1} \times d_i$,

The Matrix Chain Multiplication problem

The Problem

You are required to compute the matrix product $A_1 \cdot A_2 \cdots A_n$, where matrix A_i has dimensions $d_{i-1} \times d_i$, while minimizing the number of scalar multiplications.

The Matrix Chain Multiplication problem

The Problem

You are required to compute the matrix product $A_1 \cdot A_2 \cdots A_n$, where matrix A_i has dimensions $d_{i-1} \times d_i$, while minimizing the number of scalar multiplications.

Observe that,

The Matrix Chain Multiplication problem

The Problem

You are required to compute the matrix product $A_1 \cdot A_2 \cdots A_n$, where matrix A_i has dimensions $d_{i-1} \times d_i$, while minimizing the number of scalar multiplications.

Observe that,

1. The total number of scalar multiplications when multiplying two matrices of dimensions $p \times q$ and $q \times r$ is $p \cdot q \cdot r$.

The Matrix Chain Multiplication problem

The Problem

You are required to compute the matrix product $A_1 \cdot A_2 \cdots A_n$, where matrix A_i has dimensions $d_{i-1} \times d_i$, while minimizing the number of scalar multiplications.

Observe that,

1. The total number of scalar multiplications when multiplying two matrices of dimensions $p \times q$ and $q \times r$ is $p \cdot q \cdot r$.
2. The entries in the matrices do not affect the optimum solution.

The Matrix Chain Multiplication problem

The Problem

You are required to compute the matrix product $A_1 \cdot A_2 \cdots A_n$, where matrix A_i has dimensions $d_{i-1} \times d_i$, while minimizing the number of scalar multiplications.

Observe that,

1. The total number of scalar multiplications when multiplying two matrices of dimensions $p \times q$ and $q \times r$ is $p \cdot q \cdot r$.
2. The entries in the matrices do not affect the optimum solution.

Cost of enumerating all the orders

The Matrix Chain Multiplication problem

The Problem

You are required to compute the matrix product $A_1 \cdot A_2 \cdots A_n$, where matrix A_i has dimensions $d_{i-1} \times d_i$, while minimizing the number of scalar multiplications.

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1. The total number of scalar multiplications when multiplying two matrices of dimensions $p \times q$ and $q \times r$ is $p \cdot q \cdot r$.
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Cost of enumerating all the orders

The Matrix Chain Multiplication problem

The Problem

You are required to compute the matrix product $A_1 \cdot A_2 \cdots A_n$, where matrix A_i has dimensions $d_{i-1} \times d_i$, while minimizing the number of scalar multiplications.

Observe that,

1. The total number of scalar multiplications when multiplying two matrices of dimensions $p \times q$ and $q \times r$ is $p \cdot q \cdot r$.
2. The entries in the matrices do not affect the optimum solution.

Cost of enumerating all the orders

$$T(n) =$$

The Matrix Chain Multiplication problem

The Problem

You are required to compute the matrix product $A_1 \cdot A_2 \cdots A_n$, where matrix A_i has dimensions $d_{i-1} \times d_i$, while minimizing the number of scalar multiplications.

Observe that,

1. The total number of scalar multiplications when multiplying two matrices of dimensions $p \times q$ and $q \times r$ is $p \cdot q \cdot r$.
2. The entries in the matrices do not affect the optimum solution.

Cost of enumerating all the orders

$$T(n) = \begin{cases} 0, & \text{if } n = 0 \\ & \end{cases}$$

The Matrix Chain Multiplication problem

The Problem

You are required to compute the matrix product $A_1 \cdot A_2 \cdots A_n$, where matrix A_i has dimensions $d_{i-1} \times d_i$, while minimizing the number of scalar multiplications.

Observe that,

1. The total number of scalar multiplications when multiplying two matrices of dimensions $p \times q$ and $q \times r$ is $p \cdot q \cdot r$.
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Cost of enumerating all the orders

$$T(n) = \begin{cases} 0, & \text{if } n = 0 \\ \sum_{k=1}^{n-1} T(k) \cdot T(n-k), & \text{otherwise} \end{cases}$$

The Matrix Chain Multiplication problem

The Problem

You are required to compute the matrix product $A_1 \cdot A_2 \cdots A_n$, where matrix A_i has dimensions $d_{i-1} \times d_i$, while minimizing the number of scalar multiplications.

Observe that,

1. The total number of scalar multiplications when multiplying two matrices of dimensions $p \times q$ and $q \times r$ is $p \cdot q \cdot r$.
2. The entries in the matrices do not affect the optimum solution.

Cost of enumerating all the orders

$$T(n) = \begin{cases} 0, & \text{if } n = 0 \\ \sum_{k=1}^{n-1} T(k) \cdot T(n-k), & \text{otherwise} \end{cases}$$

Solving the recurrence gives the n^{th} ***Catalan number*** whose growth is $\Omega(\frac{4^n}{n^{\frac{3}{2}}})$.

Optimality Substructure

Optimality Substructure

Substructure

Optimality Substructure

Substructure

If somebody gave you the **first** grouping, can the problem be simplified?

Optimality Substructure

Substructure

If somebody gave you the **first** grouping, can the problem be simplified?

Yes!

Optimality Substructure

Substructure

If somebody gave you the **first** grouping, can the problem be simplified?

Yes! The two subproblems that result must be solved optimally.

Optimality Substructure

Substructure

If somebody gave you the **first** grouping, can the problem be simplified?

Yes! The two subproblems that result must be solved optimally. (Why?)

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Therefore, the optimality substructure applies.

Optimality Substructure

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Yes! The two subproblems that result must be solved optimally. (Why?)

Therefore, the optimality substructure applies.

Let $m[i, j]$ denote the optimal number of scalar multiplications to multiply the matrices $\langle A_i, A_{i+1}, \ldots A_j \rangle$.

Optimality Substructure

Substructure

If somebody gave you the **first** grouping, can the problem be simplified?

Yes! The two subproblems that result must be solved optimally. (Why?)

Therefore, the optimality substructure applies.

Let $m[i, j]$ denote the optimal number of scalar multiplications to multiply the matrices $\langle A_i, A_{i+1}, \ldots A_j \rangle$.

$$m[i, j] =$$

Optimality Substructure

Substructure

If somebody gave you the **first** grouping, can the problem be simplified?

Yes! The two subproblems that result must be solved optimally. (Why?)

Therefore, the optimality substructure applies.

Let $m[i, j]$ denote the optimal number of scalar multiplications to multiply the matrices $\langle A_i, A_{i+1}, \ldots A_j \rangle$.

$$m[i, j] = \begin{cases} 0, & \\ & \end{cases}$$

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Substructure

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Yes! The two subproblems that result must be solved optimally. (Why?)

Therefore, the optimality substructure applies.

Let $m[i,j]$ denote the optimal number of scalar multiplications to multiply the matrices $\langle A_i, A_{i+1}, \ldots A_j \rangle$.

$$m[i,j] = \begin{cases} 0, & \text{if } j = i \\ & \end{cases}$$

Optimality Substructure

Substructure

If somebody gave you the **first** grouping, can the problem be simplified?

Yes! The two subproblems that result must be solved optimally. (Why?)

Therefore, the optimality substructure applies.

Let $m[i, j]$ denote the optimal number of scalar multiplications to multiply the matrices $\langle A_i, A_{i+1}, \ldots A_j \rangle$.

$$m[i, j] = \begin{cases} 0, & \text{if } j = i \\ \min_{i \le k < j}(m[i, k] + m[k+1, j] \end{cases}$$

Optimality Substructure

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If somebody gave you the **first** grouping, can the problem be simplified?

Yes! The two subproblems that result must be solved optimally. (Why?)

Therefore, the optimality substructure applies.

Let $m[i,j]$ denote the optimal number of scalar multiplications to multiply the matrices $\langle A_i, A_{i+1}, \ldots A_j \rangle$.

$$m[i,j] = \begin{cases} 0, & \text{if } j = i \\ \min_{i \leq k < j}(m[i,k] + m[k+1,j] + d_{i-1} \cdot d_k \cdot d_j), & \end{cases}$$

Optimality Substructure

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Let $m[i,j]$ denote the optimal number of scalar multiplications to multiply the matrices $\langle A_i, A_{i+1}, \ldots A_j \rangle$.

$$m[i,j] = \begin{cases} 0, & \text{if } j = i \\ \min_{i \le k < j}(m[i,k] + m[k+1,j] + d_{i-1} \cdot d_k \cdot d_j), & \text{if } j > i. \end{cases}$$

Resource analysis

Resource analysis

Analysis

Resource analysis

Analysis

1. For space usage, observe that we need an array $m[i, j]$ and some variable space.

Resource analysis

Analysis

1. For space usage, observe that we need an array $m[i, j]$ and some variable space. Thus, space usage is $\Theta(n^2)$.

Resource analysis

Analysis

1. For space usage, observe that we need an array $m[i, j]$ and some variable space. Thus, space usage is $\Theta(n^2)$.
2. For time, note that each entry requires $O(n)$ time.

Resource analysis

Analysis

1. For space usage, observe that we need an array $m[i,j]$ and some variable space. Thus, space usage is $\Theta(n^2)$.
2. For time, note that each entry requires $O(n)$ time. Since there are $\Theta(n^2)$ entries to be filled out, the time taken by out dynamic programming algorithm is $O(n^3)$.

Resource analysis

Analysis

1. For space usage, observe that we need an array $m[i,j]$ and some variable space. Thus, space usage is $\Theta(n^2)$.
2. For time, note that each entry requires $O(n)$ time. Since there are $\Theta(n^2)$ entries to be filled out, the time taken by out dynamic programming algorithm is $O(n^3)$.

   Can you show that the time required is $\Theta(n^3)$?

Resource analysis

Analysis

1. For space usage, observe that we need an array $m[i, j]$ and some variable space. Thus, space usage is $\Theta(n^2)$.
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   Can you show that the time required is $\Theta(n^3)$?

Note

We have left out some details in the algorithm;

Resource analysis

Analysis

1. For space usage, observe that we need an array $m[i, j]$ and some variable space. Thus, space usage is $\Theta(n^2)$.
2. For time, note that each entry requires $O(n)$ time. Since there are $\Theta(n^2)$ entries to be filled out, the time taken by out dynamic programming algorithm is $O(n^3)$.

   Can you show that the time required is $\Theta(n^3)$?

Note

We have left out some details in the algorithm; such as extracting the optimal solution.

Resource analysis

Analysis

1. For space usage, observe that we need an array $m[i, j]$ and some variable space. Thus, space usage is $\Theta(n^2)$.
2. For time, note that each entry requires $O(n)$ time. Since there are $\Theta(n^2)$ entries to be filled out, the time taken by out dynamic programming algorithm is $O(n^3)$.

   Can you show that the time required is $\Theta(n^3)$?

Note

We have left out some details in the algorithm; such as extracting the optimal solution.

The technique for extracting the optimal solution is similar to the rod-cutting problem;

Resource analysis

Analysis

1. For space usage, observe that we need an array $m[i,j]$ and some variable space. Thus, space usage is $\Theta(n^2)$.
2. For time, note that each entry requires $O(n)$ time. Since there are $\Theta(n^2)$ entries to be filled out, the time taken by out dynamic programming algorithm is $O(n^3)$.

   Can you show that the time required is $\Theta(n^3)$?

Note

We have left out some details in the algorithm; such as extracting the optimal solution.

The technique for extracting the optimal solution is similar to the rod-cutting problem; keep track of the k that is optimal for $m[i,j]$.

Resource analysis

Analysis

1. For space usage, observe that we need an array $m[i, j]$ and some variable space. Thus, space usage is $\Theta(n^2)$.
2. For time, note that each entry requires $O(n)$ time. Since there are $\Theta(n^2)$ entries to be filled out, the time taken by out dynamic programming algorithm is $O(n^3)$.

   Can you show that the time required is $\Theta(n^3)$?

Note

We have left out some details in the algorithm; such as extracting the optimal solution.

The technique for extracting the optimal solution is similar to the rod-cutting problem; keep track of the k that is optimal for $m[i, j]$.

Example

Resource analysis

Analysis

1. For space usage, observe that we need an array $m[i, j]$ and some variable space. Thus, space usage is $\Theta(n^2)$.
2. For time, note that each entry requires $O(n)$ time. Since there are $\Theta(n^2)$ entries to be filled out, the time taken by out dynamic programming algorithm is $O(n^3)$.

   Can you show that the time required is $\Theta(n^3)$?

Note

We have left out some details in the algorithm; such as extracting the optimal solution.

The technique for extracting the optimal solution is similar to the rod-cutting problem; keep track of the k that is optimal for $m[i, j]$.

Example

Find the optimal parenthesization for the chain $\langle A_{7\times 10} \cdot B_{10\times 3} \cdot C_{3\times 8} \cdot D_{8\times 4}\rangle$.

The Longest Common Subsequence problem

The Longest Common Subsequence problem

The problem

The Longest Common Subsequence problem

The problem

1. You are given two subsequences of characters $X = \langle x_1, x_2, \ldots x_m \rangle$ and $Y = \langle y_1, y_2, \ldots y_n \rangle$ over an alphabet Σ.

The Longest Common Subsequence problem

The problem

1. You are given two subsequences of characters $X = \langle x_1, x_2, \ldots x_m \rangle$ and $Y = \langle y_1, y_2, \ldots y_n \rangle$ over an alphabet Σ.
2. A subsequence of a sequence is defined as as a sequence whose characters occur in the same order as the original sequence

The Longest Common Subsequence problem

The problem

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Brute-Force Approach

Assuming $m < n$, X has 2^m possible subsequences.

Optimal Substructure

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Example

Find the LCS of $X = \langle A, B, C, B, D, A, B \rangle$ and $Y = \langle B, D, C, A, B, A \rangle$.

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Find the minimum cost of packing the words into lines.

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The following equations are immediate:

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Cost structure

There are 2 key observations to make:

1. Any optimal solution on k lines consists of p (say) words on the first line and the remaining $(n-p)$ words on the remaining $(k-1)$ lines.
2. If all the words fit on one line, it is sub-optimal to break up the words into two or more lines.

We first formulate the cost function.

Let t_{ij} denote the space left over on a line, when packing the words $w_i \ldots w_j$ are packed into that line.

It is not hard to see that $t_{ij} = M - (j-i) - \sum_{k=i}^{j} l_k$.

Let $s[i,j]$ denote the packing cost of packing words w_i through w_j *in one line*.
The following equations are immediate:

$$s[i,j] \quad = \quad \left\{ \right.$$

Formulating the cost function

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The following equations are immediate:

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Formulating the cost function

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The following equations are immediate:

$$s[i,j] = \begin{cases} t_{ij}^3, & \text{if } t_{ij} \geq 0 \\ 0, & t_{ij} \geq 0 \text{ and } j = n \\ & \end{cases}$$

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The following equations are immediate:

$$s[i, j] = \begin{cases} t_{ij}^3, & \text{if } t_{ij} \geq 0 \\ 0, & t_{ij} \geq 0 \text{ and } j = n \\ \infty, & t_{ij} < 0 \end{cases} \tag{3}$$

Optimal substructure

Optimal substructure

Recursive solution

Optimal substructure

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Let $m[i, j]$ be the optimal cost of packing words w_i through w_j with word w_i starting on a fresh line.

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Hence, we are interested in $m[1, n]$.

The following recurrence is immediate:

$$m[i,j] = \begin{cases} s[i,j], & \text{if } t_{ij} \geq 0 \\ \min_{i \leq k \leq j}(s[i,k] + m[k+1,j]), & \text{otherwise} \end{cases} \tag{4}$$

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Analyze the resources of the above algorithm.

The Reachability problem

The Reachability problem

The Problem

The Reachability problem

The Problem

Given a directed, unweighted graph $G = \langle, V, E \rangle$ and a pair of vertices $s, t \in V$, is there a dipath from s to t?

Graph Exploration

Graph Exploration

The Exploration Algorithm

Function EXPLORE(G, s, t)

Graph Exploration

The Exploration Algorithm

Function $\textsc{Explore}(G, s, t)$

1: $Q = \{s\}$.

Graph Exploration

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1: $Q = \{s\}$.
2: **while** $(Q \neq \emptyset)$ **do**

Graph Exploration

The Exploration Algorithm

Function $\textsc{Explore}(G, s, t)$

$Q = \{s\}$.
while $(Q \neq \emptyset)$ **do**
    Remove a vertex u from Q.

Graph Exploration

The Exploration Algorithm

Function EXPLORE(G, s, t)

1: $Q = \{s\}$.
2: **while** $(Q \neq \emptyset)$ **do**
3: Remove a vertex u from Q. $\{$If $u = t$, then t is reachable from u.$\}$

Graph Exploration

The Exploration Algorithm

Function $\text{Explore}(G, s, t)$

1: $Q = \{s\}$.
2: **while** $(Q \neq \emptyset)$ **do**
3: Remove a vertex u from Q. $\{$If $u = t$, then t is reachable from u.$\}$
4: Mark u.

Graph Exploration

The Exploration Algorithm

Function $\textsc{Explore}(G, s, t)$

```
Q = {s}.
while (Q ≠ ∅) do
    Remove a vertex u from Q. {If u = t, then t is reachable from u.}
    Mark u.
    for (all unmarked neighbors v of u) do
```

Graph Exploration

The Exploration Algorithm

Function EXPLORE(G, s, t)

$Q = \{s\}$.
while $(Q \neq \emptyset)$ **do**
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    end for

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while (Q ≠ ∅) do
    Remove a vertex u from Q. {If u = t, then t is reachable from u.}
    Mark u.
    for (all unmarked neighbors v of u) do
        Insert v into Q.
    end for
end while
```

Algorithm 4.63: The generic algorithm

Two common search techniques

Two common search techniques

Breadth-first Search

Two common search techniques

Breadth-first Search

Implement Q as a queue.

Two common search techniques

Breadth-first Search

Implement Q as a queue.

Depth-first Search

Two common search techniques

Breadth-first Search

Implement Q as a queue.

Depth-first Search

Implement Q as a stack.

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Analysis

Two common search techniques

Breadth-first Search

Implement Q as a queue.

Depth-first Search

Implement Q as a stack.

Analysis

Both algorithms run in

Two common search techniques

Breadth-first Search

Implement Q as a queue.

Depth-first Search

Implement Q as a stack.

Analysis

Both algorithms run in $O(m + n)$ time.

Middle First Search

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The adjacency matrix of a graph with n vertices, is a $n \times n$ matrix $\mathbf{A}$ where $A_{ij} = 1$, if there is an edge from vertex i to j and 0 otherwise.

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Observation

Let A^t denote the matrix product $A \cdot A \cdots A$ (t times).

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Middle First Search

Definition

The adjacency matrix of a graph with n vertices, is a $n \times n$ matrix $\mathbf{A}$ where $A_{ij} = 1$, if there is an edge from vertex i to j and 0 otherwise.

Observation

Let A^t denote the matrix product $A \cdot A \cdots A$ (t times). Then, $(A^t)_{ij}$ is the number of paths of length t from i to j.

An Example

An Example

Example

An Example

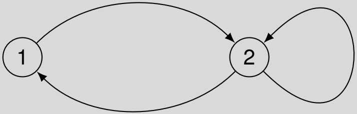
Example

Compute the matrix powers of the adjacency matrix of the following graph:

An Example

Example

Compute the matrix powers of the adjacency matrix of the following graph:



Path Theorem

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Given a graph G with n vertices and adjacency matrix $\mathbf{A}$, there is a path from s to t if and only if $(\mathbf{I} + \mathbf{A})^{n-1}_{st}$ is non-zero.

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Computing A^n - The naive approach

Path Theorem

Theorem

Given a graph G with n vertices and adjacency matrix $\mathbf{A}$*, there is a path from s to t if and only if* $(\mathbf{I} + \mathbf{A})^{n-1}_{st}$ *is non-zero.*

Computing A^n - The naive approach

Path Theorem

Theorem

Given a graph G with n vertices and adjacency matrix $\mathbf{A}$, there is a path from s to t if and only if $(\mathbf{I}+\mathbf{A})_{st}^{n-1}$ is non-zero.

Computing A^n - The naive approach

1: $\mathbf{B} = \mathbf{I}$.

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Given a graph G with n vertices and adjacency matrix $\mathbf{A}$*, there is a path from s to t if and only if* $(\mathbf{I}+\mathbf{A})^{n-1}_{st}$ *is non-zero.*

Computing A^n - The naive approach

1: $\mathbf{B} = \mathbf{I}$.
2: **for** $(i = 1$ **to** $n)$ **do**

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Given a graph G with n vertices and adjacency matrix $\mathbf{A}$*, there is a path from s to t if and only if* $(\mathbf{I}+\mathbf{A})^{n-1}_{st}$ *is non-zero.*

Computing A^n - The naive approach

1: $\mathbf{B} = \mathbf{I}$.
2: **for** $(i = 1$ **to** $n)$ **do**
3: $\quad\mathbf{B} \rightarrow \mathbf{B} \cdot (\mathbf{I}+\mathbf{A})$.

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end for

Algorithm 4.72: First approach for reachability

A smarter approach for reachability

A smarter approach for reachability

Computing A^n - The smart approach

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1: $\mathbf{B} = \mathbf{I}$.

A smarter approach for reachability

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1: $\mathbf{B} = \mathbf{I}$.
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A smarter approach for reachability

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1: $\mathbf{B} = \mathbf{I}$.
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A smarter approach for reachability

Computing A^n - The smart approach

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2: **for** ($i = 1$ **to** $\log n$) **do**
3: $\quad \mathbf{B} \rightarrow \mathbf{B} \cdot \mathbf{B}$.
4: **end for**

Algorithm 4.79: Repeated squaring

Some observations

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Observation

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1. *A multiplications step is implemented as:*

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$$B_{ij} \rightarrow \bigvee_k$$

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2. *However the case where* **B** *is a boolean matrix is sufficient for our needs! Accordingly, we can replace matrix multiplication with:*

$$B_{ij} \rightarrow \bigvee_k (B_{ik} \wedge B_{kj})$$

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3. *Strategy is called* middle-first *search, because we find to try to find a vertex k between vertices i and j.*

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$$B_{ij} \to \bigvee_k (B_{ik} \wedge B_{kj})$$

3. *Strategy is called* middle-first *search, because we find to try to find a vertex k between vertices i and j.*
4. *Strategy is inefficient in terms of time, but efficient in terms of memory.*

The All-Pairs Shortest Path problem

The All-Pairs Shortest Path problem

The Problem

The All-Pairs Shortest Path problem

The Problem

Given a weighted graph G with weights w_{ij} on edge e_{ij}

The All-Pairs Shortest Path problem

The Problem

Given a weighted graph G with weights w_{ij} on edge e_{ij} (**W**), find the length of the shortest path from vertex i to vertex j, for all pairs i and j.

The All-Pairs Shortest Path problem

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Optimality Substructure

The All-Pairs Shortest Path problem

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Optimality Substructure

Let p denote a shortest path between s and t.

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Let p denote a shortest path between s and t.

Let r be an intermediate vertex on p.

The All-Pairs Shortest Path problem

The Problem

Given a weighted graph G with weights w_{ij} on edge e_{ij} (**W**), find the length of the shortest path from vertex i to vertex j, for all pairs i and j.

Optimality Substructure

Let p denote a shortest path between s and t.

Let r be an intermediate vertex on p.

What can you say about the sub-paths of p from s to r and from r to t?

A DP based algorithm

A DP based algorithm

Shortest path algorithm

Function SHORTEST-PATHS(G, W)

A DP based algorithm

Shortest path algorithm

Function SHORTEST-PATHS(G, W)

1: $\mathbf{B} = \mathbf{W}$.

A DP based algorithm

Shortest path algorithm

Function SHORTEST-PATHS(G, W)

1: $\mathbf{B} = \mathbf{W}$.
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A DP based algorithm

Shortest path algorithm

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A DP based algorithm

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}

A DP based algorithm

Shortest path algorithm

Function SHORTEST-PATHS(G, W)

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2: **for** ($i = 1$ **to** $\log n$) **do**
3: $\quad \mathbf{B} \rightarrow \mathbf{B} \cdot \mathbf{B}$.
4: $\quad$ {The multiplication in the above step is actually implemented as:

$$B_{ij} \rightarrow \min_{k}(B_{ik} + B_{kj}).$$

$\quad$ }
5: **end for**

Algorithm 4.87: Repeated squaring for shortest paths

Iterative All-Pairs shortest path algorithm

Iterative All-Pairs shortest path algorithm

Iterative Implementation

Function SHORTEST-PATHS(G, W)

Iterative All-Pairs shortest path algorithm

Iterative Implementation

Function SHORTEST-PATHS(G, W)

1: Initialize $B_{ij}(0) = 0$, for all $i, j = 1, 2, \ldots n$.

Iterative All-Pairs shortest path algorithm

Iterative Implementation

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Iterative All-Pairs shortest path algorithm

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4:         **for** $(j = 1$ **to** $n)$ **do**

Iterative All-Pairs shortest path algorithm

Iterative Implementation

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Initialize $B_{ij}(0) = 0$, for all $i, j = 1, 2, \ldots n$.
for $(m = 1$ **to** $\log n)$ **do**
  for $(i = 1$ **to** $n)$ **do**
    for $(j = 1$ **to** $n)$ **do**
      $B_{ij}(m) = B_{ij}(m-1)$.

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4:         **for** $(j = 1$ **to** $n)$ **do**
5:             $B_{ij}(m) = B_{ij}(m-1)$.
6:             **for** $(k = 1$ **to** $n)$ **do**
7:                 $B_{ij}(m) =$

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Initialize $B_{ij}(0) = 0$, for all $i, j = 1, 2, \ldots n$.
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  for $(i = 1$ **to** $n)$ **do**
    for $(j = 1$ **to** $n)$ **do**
      $B_{ij}(m) = B_{ij}(m - 1)$.
      for $(k = 1$ **to** $n)$ **do**
        $B_{ij}(m) = \min(B_{ij}(m), B_{ik}(m - 1) + B_{kj}(m - 1))$.

Iterative All-Pairs shortest path algorithm

Iterative Implementation

Function SHORTEST-PATHS(G, W)

Initialize $B_{ij}(0) = 0$, for all $i, j = 1, 2, \ldots n$.
for $(m = 1$ **to** $\log n)$ **do**
    for $(i = 1$ **to** $n)$ **do**
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            $B_{ij}(m) = B_{ij}(m-1)$.
            for $(k = 1$ **to** $n)$ **do**
                $B_{ij}(m) = \min(B_{ij}(m), B_{ik}(m-1) + B_{kj}(m-1))$.
            end for

Iterative All-Pairs shortest path algorithm

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return(B(log n)).
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Algorithm 4.102: Implementing the shortest paths algorithm

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It follows that the algorithm takes $\Theta(n^3 \cdot \log n)$ time.

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