NP-completeness - Part II

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Outline

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Independent Set

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Query: Is there a set $V' \subseteq V$, with $|V'| \geq K$ such that for any two vertices $u, v \in V'$, $(u, v) \notin E$?

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In the graph below, $V' = \{v_2, v_4\}$ is an independent set.

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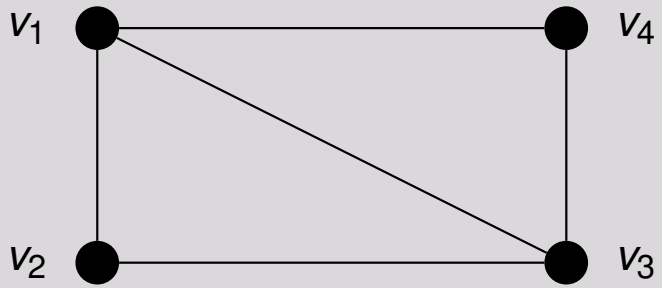

Complexity

Theorem

INDEPENDENT-SET *is* **NP-complete**.

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Proof

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   - For each one of the m clauses, we create a separate triangle in the graph.

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   - For each one of the m clauses, we create a separate triangle in the graph.
   - Each node of the triangle corresponds to a literal in the clause.

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   - For each one of the m clauses, we create a separate triangle in the graph.
   - Each node of the triangle corresponds to a literal in the clause.
   - There is an edge between two nodes u and v in different triangles if and only if $v = \neg u$.

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   - Set $K = m$.

Graphical representation

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$\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$

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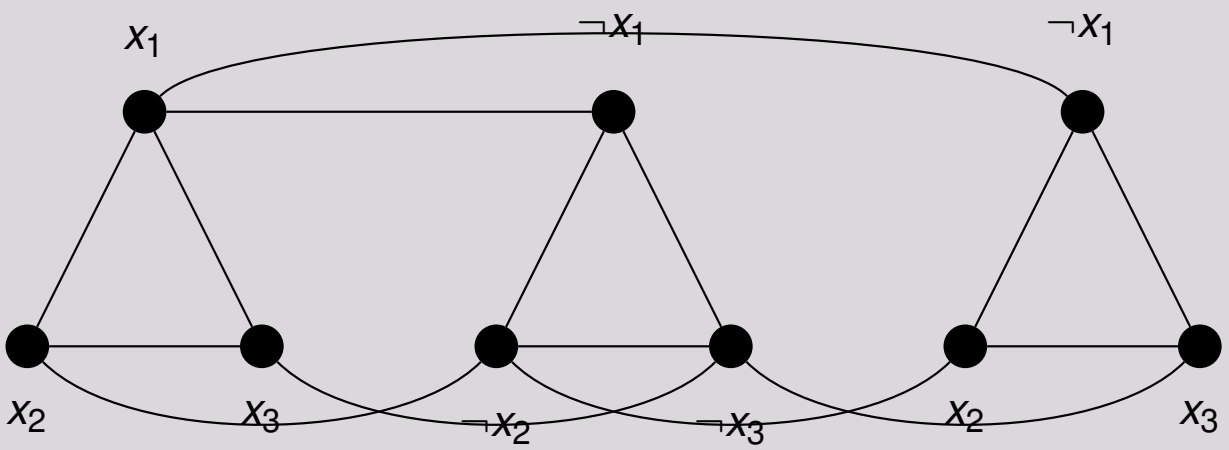

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1. Assume that a satisfying assignment exists for ϕ.
2. Pick a node in each clause triangle that is set to **true** under this assignment.

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3. The set of picked nodes must be independent. Why?

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4. We thus have an independent set of size $\geq K = m$.

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5. Now, assume that we have an independent set V' in $R(\phi)$ such that $|V'| \geq m$.

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6. Then, $|V'| = m$.

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5. Now, assume that we have an independent set V' in $R(\phi)$ such that $|V'| \geq m$.
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7. Set the literal corresponding to the vertex picked from each triangle to **true**.

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Proof

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4. We thus have an independent set of size $\geq K = m$.
5. Now, assume that we have an independent set V' in $R(\phi)$ such that $|V'| \geq m$.
6. Then, $|V'| = m$. Why?
7. Set the literal corresponding to the vertex picked from each triangle to **true**.
8. Since no pair of complementary literals is picked, the truth assignment is consistent.

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2. Pick a node in each clause triangle that is set to **true** under this assignment.
3. The set of picked nodes must be independent. Why?
4. We thus have an independent set of size $\geq K = m$.
5. Now, assume that we have an independent set V' in $R(\phi)$ such that $|V'| \geq m$.
6. Then, $|V'| = m$. Why?
7. Set the literal corresponding to the vertex picked from each triangle to **true**.
8. Since no pair of complementary literals is picked, the truth assignment is consistent.
9. One literal from each clause is set to **true** and hence all clauses are satisfied.

Vertex-Cover

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Query: Is there a set $V' \subseteq V$, with $|V'| \leq K$ such that for any two vertices $u, v \in V$, $(u, v) \in E$

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Input: An undirected graph $G = \langle V, E \rangle$ and a number $K \leq |V|$.

Query: Is there a set $V' \subseteq V$, with $|V'| \leq K$ such that for any two vertices $u, v \in V$, $(u, v) \in E \rightarrow (u \in V')$ or $v \in V'$?

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1. VERTEX-COVER is clearly in **NP**.
2. We reduce INDEPENDENT-SET to VERTEX-COVER.
3. Let $(G = \langle V, E \rangle, K)$ denote an instance of the INDEPENDENT-SET problem.

Complexity

Theorem

VERTEX-COVER *is* **NP-complete**.

Proof

1. VERTEX-COVER is clearly in **NP**.
2. We reduce INDEPENDENT-SET to VERTEX-COVER.
3. Let $(G = \langle V, E \rangle, K)$ denote an instance of the INDEPENDENT-SET problem.
4. The corresponding instance of the VERTEX-COVER problem is $(G = \langle V, E \rangle, |V| - K)$.

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4. The corresponding instance of the VERTEX-COVER problem is $(G = \langle V, E \rangle, |V| - K)$.
5. The crucial observation is that the vertex complement of a covering set must be independent and vice versa.

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2. We reduce INDEPENDENT-SET to CLIQUE.
3. Let $(G = \langle V, E \rangle, K)$ denote an instance of the INDEPENDENT-SET problem.
4. The corresponding instance of the CLIQUE problem is $(G^c = \langle V, E^c \rangle, K)$.

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2. We reduce INDEPENDENT-SET to CLIQUE.
3. Let $(G = \langle V, E \rangle, K)$ denote an instance of the INDEPENDENT-SET problem.
4. The corresponding instance of the CLIQUE problem is $(G^c = \langle V, E^c \rangle, K)$.
5. The crucial observation is that any independent set in G corresponds to a clique of the same size in G^c and vice versa.

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Query: Is there a function $f : V \to C$, such that for all $(u, v) \in E$, $f(u) \neq f(v)$?

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3. Let $\phi = C_1 \wedge C_2 \ldots C_m$ be a 3CNF formula over n variables and m clauses.

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   1. $V = \{a\}$

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4. The corresponding instance of GRAPH 3-COLORABILITY is the graph $G = \langle V, E \rangle$ constructed as follows:
   1. $V = \{a\} \cup \{x_i, \neg x_i\}, \forall i = 1, 2, \ldots, n$

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   1. $V = \{a\} \cup \{x_i, \neg x_i\}, \forall i = 1, 2, \ldots, n \cup \{C_{i1}, C_{i2}, C_{i3}\}, \forall i = 1, 2 \ldots m$, where C_{ij} refers to the j^{th} literal in the clause C_i.

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   2. $E_1 = \{a, x_i\}, \forall i = 1, 2, \ldots n$

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   2. $E_1 = \{a, x_i\}, \forall i = 1, 2, \ldots n \cup \{a, \neg x_i\}, \forall i = 1, 2, \ldots n$.
   3. $E_2 = \{C_{i1}, C_{i2}\}$

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   2. $E_1 = \{a, x_i\}, \forall i = 1, 2, \ldots n \cup \{a, \neg x_i\}, \forall i = 1, 2, \ldots n$.
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   2. $E_1 = \{a, x_i\}, \forall i = 1, 2, \ldots n \cup \{a, \neg x_i\}, \forall i = 1, 2, \ldots n$.
   3. $E_2 = \{C_{i1}, C_{i2}\} \cup \{C_{i1}, C_{i3}\} \cup \{C_{i2}, C_{i3}\}, \forall i = 1, 2, \ldots, m$.
   4. $E_3 = \cup\{C_{ij}, x_k\}, \forall j = 1, 2, 3, \forall i = 1, 2, \ldots, m, \forall k = 1, 2, \ldots n$, if $C_{ij} = x_k$.

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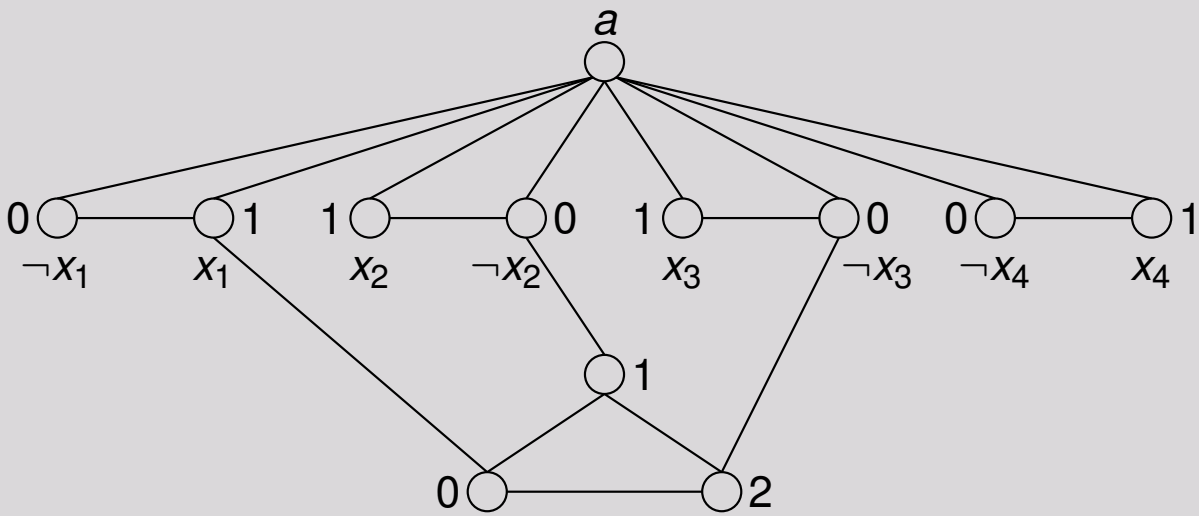

Construction for ... $\wedge$ $(x_1, \neg x_2, \neg x_3)$ $\wedge$ ...

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   The literal which is connected to a **true** literal is assigned the color 0 and the literal which is connected to a **false** literal is assigned the color 1.
   The remaining literal is assigned the color 2.

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Complexity

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   5. Set $K = 5 \cdot m$.

Example

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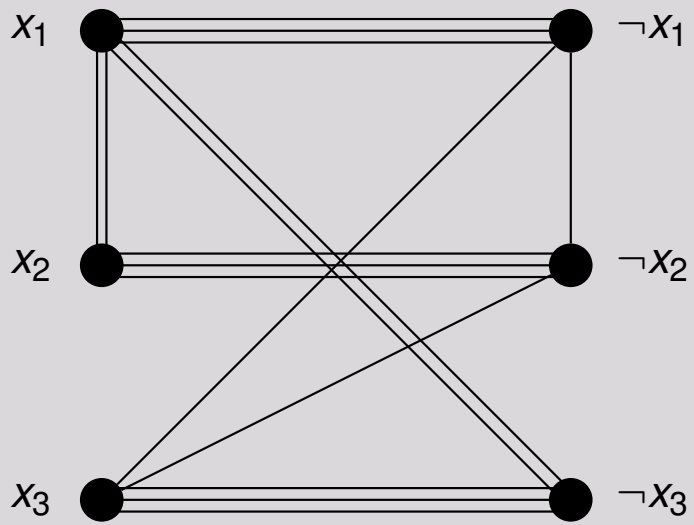

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7. Clearly, this is a consistent assignment.
8. Since each triangle is cut, it means that each clause has at least one literal set to **true** and at least one set to **false**, i.e., the assignment is nae-satisfying.

Argument - Part II

Argument - Part II

Lemma

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Assume that ϕ has a nae-satisfying assignment.

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Assume that ϕ has a nae-satisfying assignment. Then G has a cut of at least $5 \cdot m$.

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1. Let S denote the set of vertices corresponding to literals that are assigned **true**.
2. We will argue that the cut $(S, V - S)$ has at least $5 \cdot m$ edges.

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Proof

1. Let S denote the set of vertices corresponding to literals that are assigned **true**.
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2. We will argue that the cut $(S, V - S)$ has at least $5 \cdot m$ edges.
3. Since the assignment is consistent, x_i and $\neg x_i$ are on opposite sides of the cut, i.e., these vertices contribute $3 \cdot m$ edges to the cut.
4. Since the assignment is nae-satisfying, every triangle is cut and thus an additional $2 \cdot m$ edges are contributed to the cut.

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4. Since the assignment is nae-satisfying, every triangle is cut and thus an additional $2 \cdot m$ edges are contributed to the cut.
5. It follows that the cut $(S, V - S)$ has at least $5 \cdot m$ edges;

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1. Let S denote the set of vertices corresponding to literals that are assigned **true**.
2. We will argue that the cut $(S, V - S)$ has at least $5 \cdot m$ edges.
3. Since the assignment is consistent, x_i and $\neg x_i$ are on opposite sides of the cut, i.e., these vertices contribute $3 \cdot m$ edges to the cut.
4. Since the assignment is nae-satisfying, every triangle is cut and thus an additional $2 \cdot m$ edges are contributed to the cut.
5. It follows that the cut $(S, V - S)$ has at least $5 \cdot m$ edges; in fact, it has exactly $5 \cdot m$ edges.

Max-Bisection

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Definition

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Input: An undirected graph $G = \langle V, E \rangle$ and a number K.

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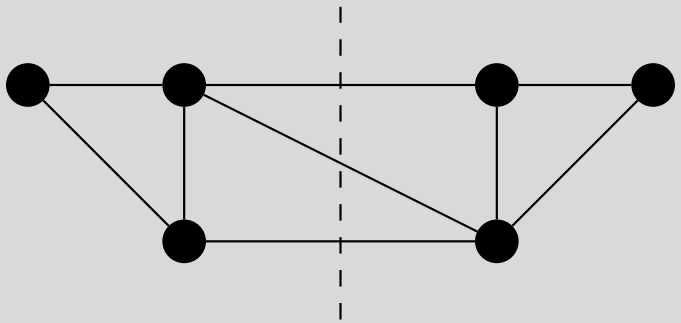
Example



Complexity

Complexity

Theorem

Complexity

Theorem

MAX-BISECTION *is* **NP-complete**.

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Proof

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2. We reduce MAXCUT to MAX-BISECTION.

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   2. $E' = E$.
   3. $K' = K$.

Argument

Argument

Completing the argument

Argument

Completing the argument

It is not hard to see that every cut in G can be made into a bisection in G' by appropriately distributing the isolated vertices.

Bisection-Width

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Definition

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Input: An undirected graph $G = \langle V, E \rangle$ and a number K.

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Input: An undirected graph $G = \langle V, E \rangle$ and a number K.

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BISECTION-WIDTH imposes an additional constraint on MINCUT, just as MAX-BISECTION imposes an additional constraint on MAXCUT.

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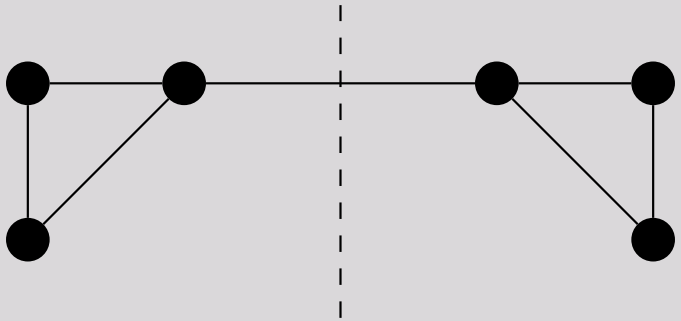
Example



Complexity

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Complexity

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Proof

1. BISECTION-WIDTH is clearly in **NP**.
2. We reduce MAX-BISECTION to BISECTION-WIDTH.
3. Let $(G = \langle V, E \rangle, K)$ denote an instance of MAX-BISECTION.

Complexity

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2. We reduce MAX-BISECTION to BISECTION-WIDTH.
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4. Without loss of generality, assume that $|V| = 2 \cdot n$.

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1. BISECTION-WIDTH is clearly in **NP**.
2. We reduce MAX-BISECTION to BISECTION-WIDTH.
3. Let $(G = \langle V, E \rangle, K)$ denote an instance of MAX-BISECTION.
4. Without loss of generality, assume that $|V| = 2 \cdot n$. Why?
5. The corresponding instance of BISECTION-WIDTH is: $(G^c = \langle V, E^c \rangle, n^2 - K)$.

Complexity

Theorem

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Proof

1. BISECTION-WIDTH is clearly in **NP**.
2. We reduce MAX-BISECTION to BISECTION-WIDTH.
3. Let $(G = \langle V, E \rangle, K)$ denote an instance of MAX-BISECTION.
4. Without loss of generality, assume that $|V| = 2 \cdot n$. Why?
5. The corresponding instance of BISECTION-WIDTH is: $(G^c = \langle V, E^c \rangle, n^2 - K)$.
6. It is not hard to see that G has a bisection of size K or more if and only if G^c has a bisection of size $n^2 - K$ or less.

Subset-Sum

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Input: A list $S = \{a_1, a_2, \ldots, a_n\}$ and a target T.

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Input: A list $S = \{a_1, a_2, \ldots, a_n\}$ and a target T.

Query: Is there a set $S' \subseteq S$, such that $\sum_{a_i \in S'} a_i = T$?

Complexity

Complexity

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Proof

1. SUBSET-SUM is clearly in **NP**.
2. We will reduce 3SAT to SUBSET-SUM.

Complexity

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SUBSET-SUM *is* **NP-complete**.

Proof

1. SUBSET-SUM is clearly in **NP**.
2. We will reduce 3SAT to SUBSET-SUM.
3. Given an instance $\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ of m clauses over n variables, we construct the following instance of SUBSET-SUM:

Complexity

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3. Given an instance $\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ of m clauses over n variables, we construct the following instance of SUBSET-SUM:
   1. We will create $2 \cdot (m + n)$ numbers, each having $(m + n)$ digits.

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   1. We will create $2 \cdot (m + n)$ numbers, each having $(m + n)$ digits.
   2. Corresponding to each variable x_i, there are two numbers T_i and F_i.

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   2. Corresponding to each variable x_i, there are two numbers T_i and F_i.
   3. Corresponding to each clause C_i, there are two rows Sl_1 and Sl_2.

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    1. We will create $2 \cdot (m + n)$ numbers, each having $(m + n)$ digits.
    2. Corresponding to each variable x_i, there are two numbers T_i and F_i.
    3. Corresponding to each clause C_i, there are two rows Sl_1 and Sl_2.
    4. Finally, we create a target which has 1 in the first n digits and 4 in the final m digits.

Example

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Let $\phi = (x_1, \neg x_3, \neg x_4) \wedge (\neg x_1, x_2, \neg x_4)$.

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The corresponding instance of SUBSET-SUM is given below:

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The corresponding instance of SUBSET-SUM is given below:

	x_1	x_2	x_3	x_4	c_1	c_2
T_1	1	0	0	0	1	0
F_1	1	0	0	0	0	1
T_2	0	1	0	0	0	1
F_2	0	1	0	0	0	0
T_3	0	0	1	0	0	0
F_3	0	0	1	0	1	0
T_4	0	0	0	1	0	0
F_4	0	0	0	1	1	1
$S1_1$	0	0	0	0	1	0
$S1_2$	0	0	0	0	2	0
$S2_1$	0	0	0	0	0	1
$S2_2$	0	0	0	0	0	2
Target	1	1	1	1	4	4

Argument

Argument

Proof

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Proof

1. Assume that ϕ is satisfiable.

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2. Pick all the rows that correspond to **true** literals.

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1. Assume that ϕ is satisfiable.
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4. Since each clause C_i is satisfied, at least one number in which $c_i = 1$ is picked.
5. Depending on whether C_i is satisfied by one literal, two literals or all three literals, we pick SI_1 and SI_2, or SI_2 or SI_1 respectively.

Argument

Proof

1. Assume that ϕ is satisfiable.
2. Pick all the rows that correspond to **true** literals.
3. Since the assignment is consistent, the first n bits of the target T are met by these n literals.
4. Since each clause C_i is satisfied, at least one number in which $c_i = 1$ is picked.
5. Depending on whether C_i is satisfied by one literal, two literals or all three literals, we pick Sl_1 and Sl_2, or Sl_2 or Sl_1 respectively.
6. Clearly the final m bits of the target are met.

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6. Clearly the final m bits of the target are met.
7. Now assume that the target T is met by some subset of numbers.

Argument

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6. Clearly the final m bits of the target are met.
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8. We must have picked exactly one of T_i and F_i for each i. Why?

Argument

Proof

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7. Now assume that the target T is met by some subset of numbers.
8. We must have picked exactly one of T_i and F_i for each i. Why?
9. If T_i is picked, set x_i to **true**; otherwise, set it to **false**.

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Proof

1. Assume that ϕ is satisfiable.
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10. We thus have a consistent assignment.

Argument

Proof

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7. Now assume that the target T is met by some subset of numbers.
8. We must have picked exactly one of T_i and F_i for each i. Why?
9. If T_i is picked, set x_i to **true**; otherwise, set it to **false**.
10. We thus have a consistent assignment.
11. Since the final m bits of the target are met, we cannot have a case where all literals of a clause are set to **false**.

Partition

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Definition

Input: A list of numbers $S = \{a_1, a_2, \ldots a_n\}$.

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Input: A list of numbers $S = \{a_1, a_2, \ldots a_n\}$.

Query: Is there a set $S' \subseteq S$, such that $\sum_{a_j \in S'} a_j = \sum_{a_j \in S - S'} a_j$?

Complexity

Complexity

Theorem

Complexity

Theorem

PARTITION *is* **NP-complete**.

Complexity

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Proof

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Proof

1. PARTITION is clearly in **NP**.

Complexity

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Proof

1. PARTITION is clearly in **NP**.
2. We reduce SUBSET-SUM to PARTITION.

Complexity

Theorem

PARTITION *is* **NP-complete**.

Proof

1. PARTITION is clearly in **NP**.
2. We reduce SUBSET-SUM to PARTITION.
3. Let $(S = \{a_1, a_2, \ldots, a_n\}, T)$ denote an instance of SUBSET-SUM.

Complexity

Theorem

PARTITION *is* **NP-complete**.

Proof

1. PARTITION is clearly in **NP**.
2. We reduce SUBSET-SUM to PARTITION.
3. Let $(S = \{a_1, a_2, \ldots, a_n\}, T)$ denote an instance of SUBSET-SUM.
4. The corresponding instance of PARTITION is:

Complexity

Theorem

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Proof

1. PARTITION is clearly in **NP**.
2. We reduce SUBSET-SUM to PARTITION.
3. Let $(S = \{a_1, a_2, \ldots, a_n\}, T)$ denote an instance of SUBSET-SUM.
4. The corresponding instance of PARTITION is:
   $R = \{a_1, a_2, \ldots, a_n, L + T, 2 \cdot L - T\}$, where $L = \sum_{a_i \in S} a_i$.

Argument

Argument

Completing the argument

Argument

Completing the argument

1. Assume that S has a subset S' which sums to T.

Argument

Completing the argument

1. Assume that S has a subset S' which sums to T.
2. We can partition the set R into the sets $S' \cup \{2 \cdot L - T\}$ and $S \setminus S' \cup \{L + T\}$.

Argument

Completing the argument

1. Assume that S has a subset S' which sums to T.
2. We can partition the set R into the sets $S' \cup \{2 \cdot L - T\}$ and $S \setminus S' \cup \{L + T\}$.
3. Both sets sum to

Argument

Completing the argument

1. Assume that S has a subset S' which sums to T.
2. We can partition the set R into the sets $S' \cup \{2 \cdot L - T\}$ and $S \setminus S' \cup \{L + T\}$.
3. Both sets sum to $2 \cdot L$!

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4. Now assume that R has a partition (R_1, R_2).

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2. We can partition the set R into the sets $S' \cup \{2 \cdot L - T\}$ and $S \setminus S' \cup \{L + T\}$.
3. Both sets sum to $2 \cdot L$!
4. Now assume that R has a partition (R_1, R_2).
5. Both R_1 and R_2 sum to $2 \cdot L$.
6. Can $L + T$ and $2 \cdot L - T$ belong to the same partition?

Argument

Completing the argument

1. Assume that S has a subset S' which sums to T.
2. We can partition the set R into the sets $S' \cup \{2 \cdot L - T\}$ and $S \setminus S' \cup \{L + T\}$.
3. Both sets sum to $2 \cdot L$!
4. Now assume that R has a partition (R_1, R_2).
5. Both R_1 and R_2 sum to $2 \cdot L$.
6. Can $L + T$ and $2 \cdot L - T$ belong to the same partition?
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6. Can $L + T$ and $2 \cdot L - T$ belong to the same partition?
7. Assume that $2 \cdot L - T \in R_1$.
8. The remaining elements in R_1 are all in S and clearly sum to T!

Knapsack

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$$\sum_{i=1}^{n} p_i \cdot x_i \quad \geq \quad P?$$

Complexity

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KNAPSACK *is* **NP-complete**.

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   2. Set $W = P = T$.

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KNAPSACK *is* **NP-complete**.

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1. KNAPSACK is clearly in **NP**.
2. We reduce SUBSET-SUM to KNAPSACK.
3. Given an instance of SUBSET-SUM, create the following instance of KNAPSACK:
   1. Set $w_i = p_i = a_i, \forall i = 1, 2, \ldots n$.
   2. Set $W = P = T$.
   3. Can you establish that the instance of SUBSET-SUM is **true** if and only if the instance of KNAPSACK is?

The Power of Integer Programming

The Power of Integer Programming

Exercise

Reduce all the problems discussed thus far to Integer Programming.

Directed Hamilton Path

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Input: A directed graph $G = \langle V, E \rangle$.

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Reduction

3SAT $\leq$ DIRECTED-HAMPATH.

$s - t$ Directed Hamilton Path

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Reduction

Same as above.

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$s - t$ DIRECTED-HAMPATH $\leq$ DIRECTED-HAMCYCLE.

Exercise

Can you provide a reduction from DIRECTED-HAMPATH *to* DIRECTED-HAMCYCLE*?*

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Input: An directed graph $G = \langle V, E \rangle$, a pairwise distance matrix **D** and a budget B.

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DIRECTED-HAMCYCLE $\leq$ TSP(D).

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Reduction

DIRECTED-HAMCYCLE $\leq \triangle$TSP(D).

Longest Path

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Input: An directed graph $G = \langle V, E, \mathbf{c} \rangle$, where $\mathbf{c} : E \to \mathcal{Z}$ is a cost function and a cost value K.

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DIRECTED-HAMCYCLE $\leq$ LONGEST-PATH.

Degree-restricted Spanning Tree

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Definition

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Input: An undirected graph $G = \langle V, E, \mathbf{c} \rangle$, where $\mathbf{c} : E \to \mathcal{Z}$ is a cost function, a degree measure D and a cost value K.

Degree-restricted Spanning Tree

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Input: An undirected graph $G = \langle V, E, \mathbf{c} \rangle$, where $\mathbf{c} : E \to \mathcal{Z}$ is a cost function, a degree measure D and a cost value K.

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Reduction

DIRECTED-HAMPATH $\leq$ DEG-SPANNING-TREE.

Exact Spanning Tree

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Reduction

SUBSET-SUM $\leq$ EXACT-SPANNING-TREE.