Computational Complexity - Homework I

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1 Instructions

1. The homework is due on February 10.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.

2 Problems

1. Let f denote a convex function. It follows that for any x_1, x_2 in the domain and $\lambda \in [0, 1]$,§

$$f(\lambda \cdot x_1 + (1 - \lambda) \cdot x_2) \leq \lambda \cdot f(x_1) + (1 - \lambda) \cdot f(x_2).$$

Let X denote a random variable. Jensen's inequality states that

$$\mathbf{E}[f(X)] \geq f(\mathbf{E}[X])$$

Prove the above inequality for the case when X takes on precisely two values x_1 and x_2 with probabilities p and $(1 - p)$ respectively. Argue that Jensen's inequality becomes an equality, when $f(X) = a \cdot X + b$, where a and b are constants.

2. Let $E_1, E_2, \ldots E_n$ denote a set of events. Argue that

$$\mathbf{P}[\overline{\cup_{i=1}^{n} E_i}] = \sum_{T \subseteq \{1,2,\ldots,n\}} (-1)^{|T|} \mathbf{P}[\cap_{i \in T} E_i]$$

3. Prove that in any finite graph, the number of vertices with odd degrees is even,

4. Consider two variants of the Hamilton cycle problem: In Variant I, you are required to provide a "yes/no" answer to the question: Does the graph G have a Hamilton cycle? In Variant II, you are required to actually provide the Hamilton cycle in G, if one exists. Assume that an oracle for Variant I exists. Argue that by querying this oracle at most a polynomial number of times (polynomial in the size of G), we can solve Variant II.

5. Consider the Fibonacci series, which is defined as follows:

$$\begin{aligned} F(1) &= 1 \\ F(2) &= 1 \\ F(n) &= F(n-1) + F(n-2), n \geq 3 \end{aligned}$$

Argue that $F(n) = \Theta(\psi^n)$, where $\psi = \frac{1+\sqrt{5}}{2}$.
What is the complexity of checking whether a given number n is a Fibonacci number?