Quadratic Programming: Applications

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Outline

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1. Mean-Variance Optimization
2. Brief mention of other MVO models

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Markowitz' theory of mean-variance optimization

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Mechanism for the selection of portfolios

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- For $i \neq j$, ρ_{ij} denotes the correlation coefficient of the returns of assets S_i and S_j.
- Let $\boldsymbol{\mu} = [\mu_1, \cdots, \mu_n]^T$, and $\boldsymbol{\Sigma} = (\sigma_{ij})$ be the $n \times n$ symmetric covariance matrix with $\sigma_{ii} = \sigma_i^2$ and $\sigma_{ij} = \rho_{ij} \cdot \sigma_i \cdot \sigma_j$ for $i \neq j$.

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- Since variance is always nonnegative, it follows that $\mathbf{x}^T \cdot \mathbf{\Sigma} \cdot \mathbf{x} \geq \mathbf{0}$ for any $\mathbf{x}$, i.e., $\mathbf{\Sigma}$ is positive semidefinite.

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- In particular, one of the constraints in the set $\mathcal{X}$ is:

$$\sum_{i=1}^{n} x_i = 1.$$

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- By solving this problem for values of $\mathbf{R}$ ranging between $\mathbf{R}_{min}$ and $\mathbf{R}_{max}$, we obtain all efficient portfolios.

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- The annual times series for the "total return" for each asset between 1960 and 2003 are given in the next table.

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Markowitz theory of mean-variance optimization

Table: Total returns for stocks, bonds, and money market

Markowitz theory of mean-variance optimization

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Year	Stocks	Bonds	MM	Year	Stocks	Bonds	MM
1960	20.255	262.935	100.00	1982	115.308	777.332	440.68
1961	25.686	268.730	102.33	1983	141.316	787.357	482.42
1962	23.430	284.090	105.33	1984	150.181	907.712	522.84
1963	28.746	289.162	108.89	1985	197.829	1200.63	566.08
1964	33.448	299.894	113.08	1986	234.755	1469.45	605.20
1965	37.581	302.695	117.97	1987	247.080	1424.91	646.17
1966	33.784	318.197	124.34	1988	288.116	1522.40	702.77
1967	41.873	309.103	129.94	1989	379.409	1804.63	762.16
1968	46.480	316.051	137.77	1990	367.636	1944.25	817.87
1969	42.545	298.249	150.12	1991	479.633	2320.64	854.10
1970	44.221	354.671	157.48	1992	516.178	2490.97	879.04
1971	50.545	394.532	164.00	1993	568.202	2816.40	905.06
1972	60.146	403.942	172.74	1994	575.705	2610.12	954.39
1973	51.311	417.252	189.93	1995	792.042	3287.27	1007.84
1974	37.731	433.927	206.13	1996	973.897	3291.58	1061.15
1975	51.777	457.885	216.85	1997	1298.82	3687.33	1119.51
1976	64.166	529.141	226.93	1998	1670.01	4220.24	1171.91
1977	59.574	531.144	241.82	1999	2021.40	3903.32	1234.02
1978	63.488	524.435	266.07	2000	1837.36	4575.33	1313.00
1979	75.303	531.040	302.74	2001	1618.98	4827.26	1336.89
1980	99.780	517.860	359.96	2002	1261.18	5558.40	1353.47
1981	94.867	538.769	404.48	2003	1622.94	5588.19	1366.73

Markowitz theory of mean-variance optimization

Markowitz theory of mean-variance optimization

Calculating rates of return

Markowitz theory of mean-variance optimization

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Markowitz theory of mean-variance optimization

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Markowitz theory of mean-variance optimization

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Year	Stocks	Bonds	MM	Year	Stocks	Bonds	MM
1961	26.810	2.200	2.33	1983	22.560	1.290	9.47
1962	-8.780	5.720	2.93	1984	6.270	15.290	8.38
1963	22.690	1.790	3.38	1985	31.170	32.270	8.27
1964	16.360	3.710	3.85	1986	18.670	22.39	6.91
1965	12.360	0.930	4.32	1987	5.250	-3.03	6.77
1966	-10.100	5.120	5.40	1988	16.610	6.84	8.76
1967	23.940	-2.860	4.51	1989	31.690	18.54	8.45
1968	11.000	2.250	6.02	1990	-3.100	7.74	7.31
1969	-8.470	-5.630	8.97	1991	30.460	19.36	4.43
1970	3.940	18.920	4.90	1992	7.620	7.34	2.92
1971	14.300	11.240	4.14	1993	10.080	13.06	2.96
1972	18.990	2.390	5.33	1994	1.320	-7.32	5.45
1973	-14.690	3.290	9.95	1995	37.580	25.94	5.60
1974	-26.470	4.000	8.53	1996	22.960	0.13	5.29
1975	37.230	5.520	5.20	1997	33.360	12.02	5.50
1976	23.930	15.560	4.65	1998	28.58	14.45	4.68
1977	-7.160	0.380	6.56	1999	21.04	-7.51	5.30
1978	6.570	-1.260	10.03	2000	-9.10	17.22	6.40
1979	18.610	-1.260	13.78	2001	-11.89	5.51	1.82
1980	32.500	-2.480	18.90	2002	-22.10	15.15	1.24
1981	-4.920	4.040	12.37	2003	28.68	0.54	0.98
1982	21.550	44.280	8.95				

Markowitz theory of mean-variance optimization

Markowitz theory of mean-variance optimization

Arithmetic mean

Markowitz theory of mean-variance optimization

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Markowitz theory of mean-variance optimization

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Markowitz theory of mean-variance optimization

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- This gives:

	Stocks	Bonds	MM
Arithmetic mean $\bar{r}_i$	12.06%	7.85%	6.32%

Markowitz theory of mean-variance optimization

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Geometric mean

Markowitz theory of mean-variance optimization

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Markowitz theory of mean-variance optimization

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	Stocks	Bonds	MM
Geometric mean μ_i	10.73%	7.37%	6.27%

Markowitz theory of mean-variance optimization

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Covariance matrix

Markowitz theory of mean-variance optimization

Covariance matrix

- We also compute the covariance matrix:

Markowitz theory of mean-variance optimization

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$$cov(R_i, R_j) = \frac{1}{T}\sum_{i=1}^{T}(r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j)$$

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Covariance	Stocks	Bonds	MM
Stocks	0.02778	0.00387	0.00021
Bonds	0.00387	0.01112	-0.00020
MM	0.00021	-0.00020	0.00115

Markowitz theory of mean-variance optimization

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Volatility of the rate of return

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	Stocks	Bonds	MM
Volatility	16.67%	10.55%	3.40%

Markowitz theory of mean-variance optimization

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Correlation matrix

Markowitz theory of mean-variance optimization

Correlation matrix

- The correlation matrix is computed using:

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- This gives:

Correlation	Stocks	Bonds	MM
Stocks	1	0.2199	0.0366
Bonds	0.2199	1	-0.0545
MM	0.0366	-0.0545	1

Markowitz theory of mean-variance optimization

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The quadratic program for this problem is as follows:

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$$\min \ \frac{1}{2} \cdot [0.02778 \cdot x_S^2 + 2 \cdot 0.00387 \cdot x_S \cdot x_B + 2 \cdot 0.00021 \cdot x_S \cdot x_M + 0.01112 \cdot x_B^2 - 2 \cdot 0.00020 \cdot x_B \cdot x_M + 0.00115 \cdot x_M^2]$$

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- Solving for $R = 6.5\%$ to $R = 10.5\%$ with increments of 0.5%, gives us the optimal portfolios shown in the next table.

Markowitz theory of mean-variance optimization

Markowitz theory of mean-variance optimization

Table of efficient portfolios

Markowitz theory of mean-variance optimization

Table of efficient portfolios

Rate of return R	Variance	Stocks	Bonds	MM
0.065	0.0010	0.03	0.10	0.87
0.070	0.0014	0.13	0.12	0.75
0.075	0.0026	0.24	0.14	0.62
0.080	0.0044	0.35	0.16	0.49
0.085	0.0070	0.45	0.18	0.37
0.090	0.0102	0.56	0.20	0.24
0.095	0.0142	0.67	0.22	0.11
0.100	0.0189	0.78	0.22	0.00
0.105	0.0246	0.93	0.07	0.00

Large-scale portfolio optimization

Large-scale portfolio optimization

Issues with large-scale portfolios

Large-scale portfolio optimization

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- We will consider practical issues that arise when the mean-variance model is used to construct a portfolio from a large underlying family of assets.

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Large-scale portfolio optimization

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- Let us consider a portfolio of stocks constructed from a set of n stocks with known expected returns and covariance matrix,

Large-scale portfolio optimization

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- We will consider practical issues that arise when the mean-variance model is used to construct a portfolio from a large underlying family of assets.

Example

- Let us consider a portfolio of stocks constructed from a set of n stocks with known expected returns and covariance matrix, where n may be in the hundreds or thousands.

Large-scale portfolio optimization

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Diversification

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- Practitioners often use additional constraints on the x_is to insure themselves against estimation and model errors, and to ensure that the chosen portfolio is well diversified.
  - For example, a limit m may be imposed on the size of each x_i, say $x_i \leq m$ for $i = 1, \cdots, n$.

Large-scale portfolio optimization

Large-scale portfolio optimization

Diversification

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Large-scale portfolio optimization

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- Note that the more constraints one adds to a model, the more the objective value deteriorates.

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- Note that the more constraints one adds to a model, the more the objective value deteriorates.
- So, this approach to producing diversification can be quite costly.

Large-scale portfolio optimization

Large-scale portfolio optimization

Transaction costs

Large-scale portfolio optimization

Transaction costs

- We can add a portfolio turnover constraint to ensure that the change between the current holdings x^0 and the desired portfolio x is bounded by h.

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$$x_i - x_i^0 \leq y_i, \quad y_i \geq 0$$
$$x_i^0 - x_i \leq z_i, \quad z_i \geq 0$$
$$\sum_{i=1}^{n}(y_i + z_i) \leq h$$

Large-scale portfolio optimization

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Transaction costs

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Large-scale portfolio optimization

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Large-scale portfolio optimization

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- Another interesting approach is the Black-Litterman model, which allows investors to combine their unique views regarding the performance of various assets with the market equilibrium in a manner that results in intuitive, diversified portfolios.

The Black-Litterman Model

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  - Note that the factor τ should be small since the variance $\tau \cdot \sigma_i^2$ of the random variable μ_i is typically much smaller than the variance σ_i^2 of the underlying asset returns.

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Example (to illustrate the Black-Litterman approach)

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- Applying our formula to compute $\bar{\mu}$ gives:

	Stocks	Bonds	MM
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$$\min \ \frac{1}{2} \cdot [0.02778 \cdot x_S^2 + 2 \cdot 0.00387 \cdot x_S \cdot x_B + 2 \cdot 0.00021 \cdot x_S \cdot x_M$$
$$+0.01112 \cdot x_B^2 - 2 \cdot 0.00020 \cdot x_B \cdot x_M + 0.00115 \cdot x_M^2]$$

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+0.01112 \cdot x_B^2 - 2 \cdot 0.00020 \cdot x_B \cdot x_M + 0.00115 \cdot x_M^2] & \\
0.1177 \cdot x_S + 0.0751 \cdot x_B + 0.0234 \cdot x_M & \geq R \\
x_S + x_B + x_M & = 1 \\
x_S, X_B, X_M & \geq 0
\end{aligned}
$$

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Black-Litterman efficient portfolios

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The Black-Litterman Model

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Rate of return R	Variance	Stocks	Bonds	MM
0.040	0.0012	0.08	0.17	0.75
0.045	0.0015	0.11	0.21	0.68
0.050	0.0020	0.15	0.24	0.61
0.055	0.0025	0.18	0.28	0.54
0.060	0.0032	0.22	0.31	0.47
0.065	0.0039	0.25	0.35	0.40
0.070	0.0048	0.28	0.39	0.33
0.075	0.0059	0.32	0.42	0.26
0.080	0.0070	0.35	0.46	0.19
0.085	0.0083	0.38	0.49	0.13
0.090	0.0096	0.42	0.53	0.05
0.095	0.0111	0.47	0.53	0.00
0.100	0.0133	0.58	0.42	0.00
0.105	0.0163	0.70	0.30	0.00
0.110	0.0202	0.82	0.18	0.00
0.115	0.0249	0.94	0.06	0.00

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- Generally, the greater the value of the Sharpe ratio, the more attractive the risk adjusted return.

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Capital Allocation Line (CAL)

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Characterization of the Complete Portfolio

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- Variance

  $\sigma_C{}^2 = y^2 \cdot \sigma_p{}^2 + (1 - y)^2 \cdot 0 + 2 \cdot y \cdot (1 - y) \cdot cov(r_p, r_f)$

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Capital Allocation Line (CAL)

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Available Complete Portfolios

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Available Complete Portfolios

- Solve for y

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Capital Allocation Line (CAL)

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- Replace in the equation for the expected rate of return

Capital Allocation Line (CAL)

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Available Complete Portfolios

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Capital Allocation Line (CAL)

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- Slope of CAL (Sharpe Ratio): $[E(r_p) - r_f]/\sigma_p$ or $[\boldsymbol{\mu}^T \cdot \mathbf{x} - r_f]/(\mathbf{x}^\mathbf{T} \cdot \boldsymbol{\Sigma} \cdot \mathbf{x})^{1/2}$

Capital Allocation Line (CAL)

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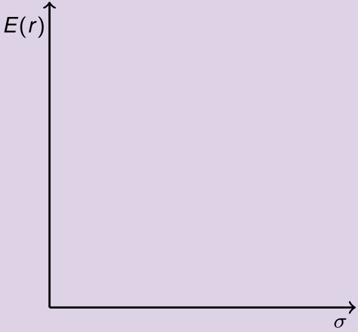

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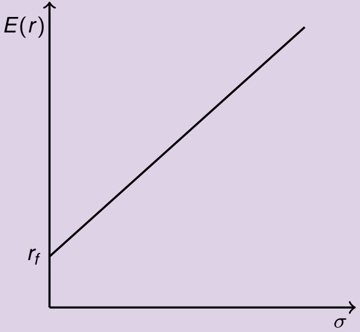

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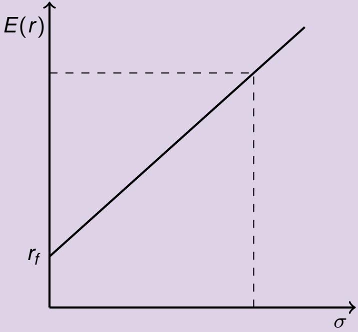

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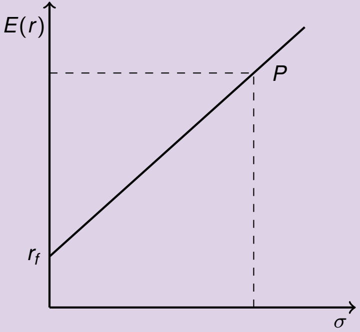

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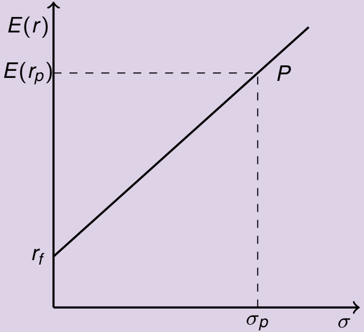

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  where $\mathbf{x_R}$ denotes the unique solution of MVO problem.
- Since we assumed that $\mathbf{\Sigma}$ is positive definite, it is easy to show that the function $\sigma(R)$ is strictly convex in its domain.

Efficient Frontier

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- Recall that *Markowitz Efficient Frontier* is the set of all portfolios of which expected returns reach the maximum given a certain level of risk.
- We denote with R_{min} and R_{max} the minimum and maximum expected returns for efficient portfolios.
- Define the function

$$\sigma(R) : [R_{min}, R_{max}] \to \mathbb{R}, \sigma(R) := (\mathbf{x_R}^T \cdot \mathbf{\Sigma} \cdot \mathbf{x_R})^{1/2},$$

  where $\mathbf{x_R}$ denotes the unique solution of MVO problem.
- Since we assumed that $\mathbf{\Sigma}$ is positive definite, it is easy to show that the function $\sigma(R)$ is strictly convex in its domain.
- We will assume that $r_f < R_{min}$, which is natural since the portfolio $\mathbf{x_{min}}$ has a positive risk associated with it while the risk-free asset does not.

Maximize the Sharpe Ratio

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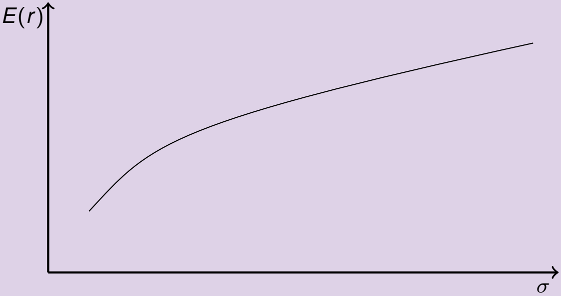

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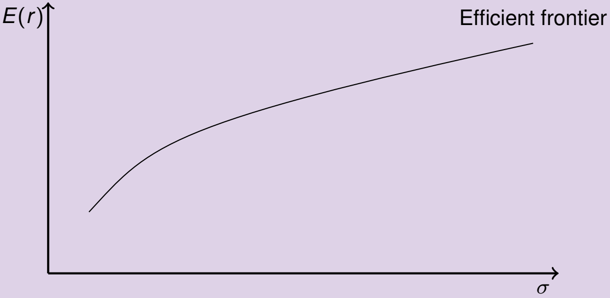

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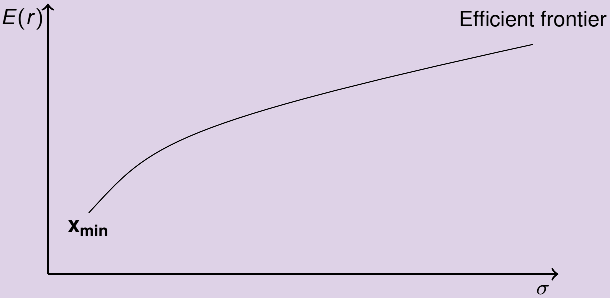

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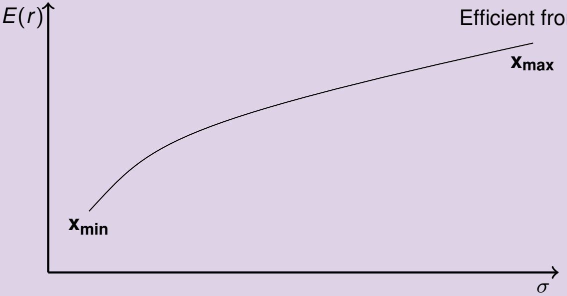

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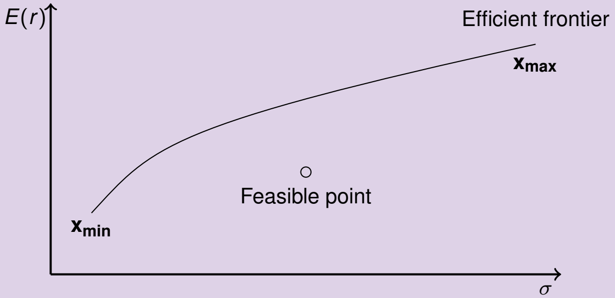

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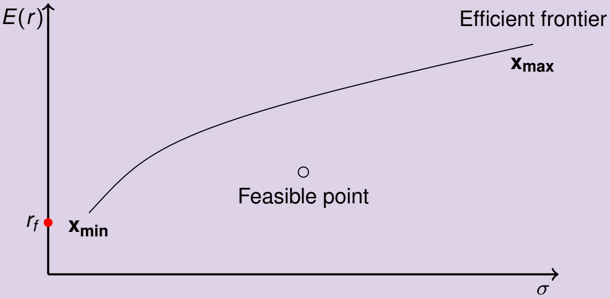

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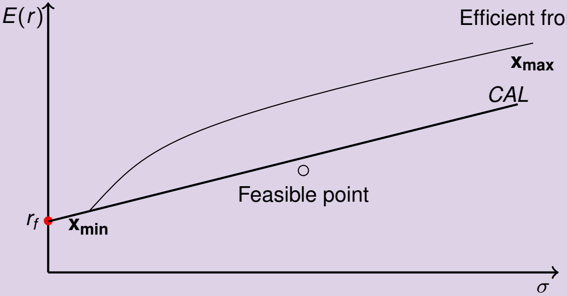

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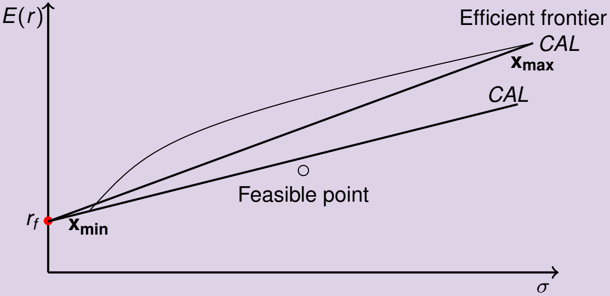

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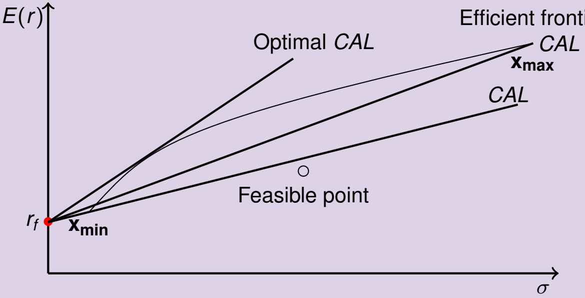

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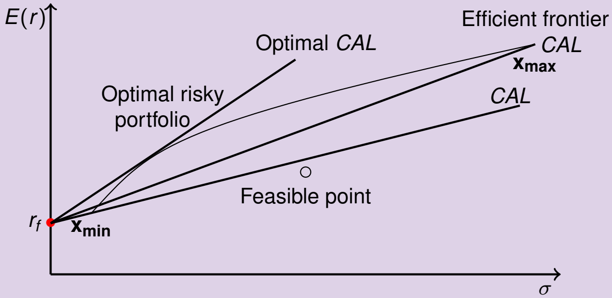

The Optimal Risky Portfolio

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- Although it has a nice polyhedral feasible region, its objective function is somewhat complicated and possibly non-concave.

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- Although it has a nice polyhedral feasible region, its objective function is somewhat complicated and possibly non-concave.
- So it is not a convex optimization problem.

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Equivalent Quadratic Programming

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1. We assume that $\sum_{i=1}^{n} x_i = 1$ for any feasible portfolio $\mathbf{x}$.

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   This is a natural assumption since the x_is are the proportions of the portfolio in different asset classes.

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   If all feasible portfolios have expected return bounded by the risk-free rate, there is no need to optimize, the risk-free investment dominates all others.

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Given a set $\mathcal{X}$ of feasible portfolios with the properties that $\mathbf{e}^{\mathbf{T}} \cdot \mathbf{x} = 1, \forall \mathbf{x} \in \mathcal{X}$ and $\exists \hat{\mathbf{x}} \in \mathcal{X}$ such that $\boldsymbol{\mu}^{\mathbf{T}} \cdot \hat{\mathbf{x}} > r_f$,

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Remark

- This is a quadratic program and can be solved by IPMs.

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Then, $\sqrt{\mathbf{x}^\mathbf{T} \cdot \boldsymbol{\Sigma} \cdot \mathbf{x}} = \frac{1}{\kappa} \cdot \sqrt{\mathbf{y}^\mathbf{T} \cdot \boldsymbol{\Sigma} \cdot \mathbf{y}}$ and the objective function can be written as $1/\sqrt{\mathbf{y}^\mathbf{T} \cdot \boldsymbol{\Sigma} \cdot \mathbf{y}}$ in terms of the new variables.

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Proof of Proposition (cont'd.)

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Exercise

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$$\mathcal{X}^+ = \{(\mathbf{x}, \kappa) \mid \mathbf{A} \cdot \mathbf{x} - \mathbf{b} \cdot \kappa \geq 0,\ \mathbf{C} \cdot \mathbf{x} - \mathbf{d} \cdot \kappa = 0,\ \kappa \geq 0\}.$$

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Assume that the risk-free return rate is 3%.

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And the geometric mean is given as

	Stocks	Bonds	MM
Geometric mean μ_i	10.73%	7.37%	6.27%

Also, the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and $b = 1$.
Assume that the risk-free return rate is 3%. Find the program of optimal risky portfolio and the equivalent quadratic programming problem.

The Optimal Risky Portfolio

The Optimal Risky Portfolio

Solution

The Optimal Risky Portfolio

Solution

$$\mathbf{\Sigma} = \begin{bmatrix} 0.02778 & 0.00387 & 0.00021 \\ 0.00387 & 0.01112 & -0.00020 \\ 0.00021 & -0.00020 & 0.00115 \end{bmatrix}$$

The Optimal Risky Portfolio

Solution

$$\mathbf{\Sigma} = \begin{bmatrix} 0.02778 & 0.00387 & 0.00021 \\ 0.00387 & 0.01112 & -0.00020 \\ 0.00021 & -0.00020 & 0.00115 \end{bmatrix}$$

$$\boldsymbol{\mu} = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix}$$

The Optimal Risky Portfolio

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The Optimal Risky Portfolio

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$$\boldsymbol{\mu} = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix}$$

So the program of optimal risky portfolio is

$$\max \quad \frac{0.1073 \cdot x_S + 0.0737 \cdot x_B + 0.0627 \cdot x_M - 0.03}{\begin{pmatrix} 0.02778 \cdot x_S^2 + 2 \cdot 0.00387 \cdot x_S \cdot x_B + 2 \cdot 0.00021 \cdot x_S \cdot x_M \\ +0.01112 \cdot x_B^2 - 2 \cdot 0.00020 \cdot x_B \cdot x_M + 0.00115 \cdot x_M^2 \end{pmatrix}^{1/2}}$$

The Optimal Risky Portfolio

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$$x_S + x_B + x_M = 1.$$

The Optimal Risky Portfolio

The Optimal Risky Portfolio

Solution

The Optimal Risky Portfolio

Solution

$$\boldsymbol{\mu} - r_f \cdot \mathbf{e} = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} - \begin{bmatrix} 0.03 \\ 0.03 \\ 0.03 \end{bmatrix}$$

The Optimal Risky Portfolio

Solution

$$\boldsymbol{\mu} - r_f \cdot \mathbf{e} = \begin{bmatrix} 0.1073 \\ 0.0737 \\ 0.0627 \end{bmatrix} - \begin{bmatrix} 0.03 \\ 0.03 \\ 0.03 \end{bmatrix} = \begin{bmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{bmatrix}$$

The Optimal Risky Portfolio

Solution

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We can find κ and $\mathbf{y}$ as below.

The Optimal Risky Portfolio

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We can find κ and $\mathbf{y}$ as below.

$$\kappa = \frac{1}{\left(\begin{bmatrix} 0.0773 \\ 0.0437 \\ 0.0327 \end{bmatrix} \right)^T \cdot \begin{bmatrix} x_S \\ x_B \\ x_M \end{bmatrix}}$$

The Optimal Risky Portfolio

Solution

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The Optimal Risky Portfolio

Solution

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$$\mathbf{y} = \begin{bmatrix} y_S \\ y_B \\ y_M \end{bmatrix}$$

The Optimal Risky Portfolio

Solution

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$$\mathbf{y} = \begin{bmatrix} y_S \\ y_B \\ y_M \end{bmatrix} = \begin{bmatrix} \frac{x_S}{0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M} \\ \frac{x_B}{0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M} \\ \frac{X_M}{0.0773 \cdot x_S + 0.0437 \cdot x_B + 0.0327 \cdot x_M} \end{bmatrix}$$

The Optimal Risky Portfolio

The Optimal Risky Portfolio

Solution

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And the equivalent quadratic programming problem is

The Optimal Risky Portfolio

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$$\begin{aligned} \min \quad & 0.02778 \cdot y_S^2 + 2 \cdot 0.00387 \cdot y_S \cdot y_B + 2 \cdot 0.00021 \cdot y_S \cdot y_M \\ & +0.01112 \cdot y_B^2 - 2 \cdot 0.00020 \cdot y_B \cdot y_M + 0.00115 \cdot y_M^2 \end{aligned}$$

The Optimal Risky Portfolio

Solution

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$$\begin{aligned} \min \quad & 0.02778 \cdot y_S^2 + 2 \cdot 0.00387 \cdot y_S \cdot y_B + 2 \cdot 0.00021 \cdot y_S \cdot y_M \\ & +0.01112 \cdot y_B^2 - 2 \cdot 0.00020 \cdot y_B \cdot y_M + 0.00115 \cdot y_M^2 \\ & \frac{y_S}{\kappa} + \frac{y_B}{\kappa} + \frac{y_M}{\kappa} = 1 \; (or \; y_S + y_B + y_M - \kappa = 0) \end{aligned}$$

The Optimal Risky Portfolio

Solution

And the equivalent quadratic programming problem is

$$\begin{aligned} \min \quad & 0.02778 \cdot y_S^2 + 2 \cdot 0.00387 \cdot y_S \cdot y_B + 2 \cdot 0.00021 \cdot y_S \cdot y_M \\ & +0.01112 \cdot y_B^2 - 2 \cdot 0.00020 \cdot y_B \cdot y_M + 0.00115 \cdot y_M^2 \\ & \frac{y_S}{\kappa} + \frac{y_B}{\kappa} + \frac{y_M}{\kappa} = 1 \; (or\; y_S + y_B + y_M - \kappa = 0) \\ & 0.0773 \cdot y_S + 0.0437 \cdot y_B + 0.0327 \cdot y_M = 1 \end{aligned}$$

The Optimal Risky Portfolio

Solution

And the equivalent quadratic programming problem is

$$
\begin{aligned}
\min \quad & 0.02778 \cdot y_S^2 + 2 \cdot 0.00387 \cdot y_S \cdot y_B + 2 \cdot 0.00021 \cdot y_S \cdot y_M \\
& +0.01112 \cdot y_B^2 - 2 \cdot 0.00020 \cdot y_B \cdot y_M + 0.00115 \cdot y_M^2 \\
& \frac{y_S}{\kappa} + \frac{y_B}{\kappa} + \frac{y_M}{\kappa} = 1 \; (or \; y_S + y_B + y_M - \kappa = 0) \\
& 0.0773 \cdot y_S + 0.0437 \cdot y_B + 0.0327 \cdot y_M = 1 \\
& \mathbf{y}, \kappa \geq 0
\end{aligned}
$$

Topics not covered

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- Returns-Based Style Analysis

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- Returns-Based Style Analysis
- Recovering Risk-Neutral Probabilities from Options Prices

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