Quadratic Programming: Theory and Algorithms

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Outline

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- You are allowed to play this game one time.
- Which game would you choose to play?

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- We buy a stock at some dollar amount per share in the beginning of the month, and sell it at some dollar amount per share at the end of the month.
- The rate of return of each stock is a random variable with some expected value.
- Our goal is to invest in such a way that the expected end-of-month return is at least \$50.00 or 5%.

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- Infer the actual decision from the solution to the mathematical model.

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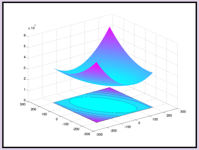

Figure: Graph and contours of a convex function

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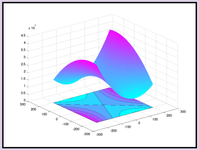

Figure: Graph and contours of a nonconvex function

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- Note that, unlike the case of linear programming, the variables of the primal quadratic programming problem also appear in the dual QP.

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Karush-Kuhn-Tucker Theorem (as applied to the QP problem)

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- *Futhermore,* $\mathbf{x}$ *is a global optimal solution.*

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- In other words, all 5 conditions are both necessary and sufficient for **x**, **y**, and **s** to describe a global optimal solution of the QP problem.

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  2. dual feasibility: $\mathbf{A}^T \cdot \mathbf{y} - \mathbf{Q} \cdot \mathbf{x} + \mathbf{s} = \mathbf{c}$, $\mathbf{s} \geq \mathbf{0}$;

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  3. complementary slackness: for each $i = 1, \cdots, n$ we have $\mathbf{x}_i \cdot \mathbf{s}_i = \mathbf{0}$.

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$$\begin{aligned} \min \quad & 2 \cdot x_1^2 + x_2^2 + 4 \cdot x_3^2 \\ & x_1 + 2 \cdot x_2 - x_3 \quad = \quad 6 \end{aligned}$$

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\min \quad & 2 \cdot x_1^2 + x_2^2 + 4 \cdot x_3^2 \\
& x_1 + 2 \cdot x_2 - x_3 = 6 \\
& 2 \cdot x_1 - 2 \cdot x_2 + 3 \cdot x_3 = 12
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 & 2 \cdot x_1 - 2 \cdot x_2 + 3 \cdot x_3 & = & 12 \\
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Karush-Kuhn-Tucker (KKT) Optimality Conditions

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- Set up the KKT conditions for the optimal solution in matrix form, and show how you would solve for $\mathbf{x}$ and $\mathbf{y}$.

Karush-Kuhn-Tucker (KKT) Optimality Conditions

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Karush-Kuhn-Tucker (KKT) Optimality Conditions

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Karush-Kuhn-Tucker (KKT) Optimality Conditions

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- After solving the system, we find that $\mathbf{x} = (5.045, 1.194, 1.433)$ is an optimal solution

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- After solving the system, we find that $\mathbf{x} = (5.045, 1.194, 1.433)$ is an optimal solution with $\mathbf{y} = (7.522, 6.328)$ and $\mathbf{s} = (0, 0, 0)$.

Interior-Point Methods

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Interior-Point Method(IPM) finds primal-dual solutions $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ by applying variants of Newton's method to the optimality conditions and modifying the search directions and step lengths so that $\mathbf{x} \geq 0$ and $\mathbf{s} \geq 0$ are satisfied strictly at every iteration.

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Rewrite the Optimality Conditions

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$$\mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{s}) = \begin{bmatrix} \mathbf{A}^T\mathbf{y} - \mathbf{Q}\mathbf{x} + \mathbf{s} - \mathbf{c} \\ \mathbf{A}\mathbf{x} - \mathbf{b} \\ \mathbf{X}\mathbf{S}\mathbf{e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\mathbf{x}, \mathbf{s}) \geq 0$$

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$\mathbf{X}$ and $\mathbf{S}$ are diagonal matrices such that $\mathbf{X}_{ii} = x_i$ and $\mathbf{X}_{ij} = 0, i \neq j$, and similarly for $\mathbf{S}$.

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1. Identify an initial solution $(\mathbf{x^0}, \mathbf{y^0}, \mathbf{s^0})$, which satisfies the first two constraints (linear) and $(\mathbf{x^0}, \mathbf{s^0}) > 0$, but not the third one.

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2. Generate new points $(\mathbf{x^k}, \mathbf{y^k}, \mathbf{s^k})$ that also satisfy these same conditions and get progressively closer to satisfying the third constraint.

Algorithms for IPMs with pure Newton direction

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Definition

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- Feasible set: $\mathcal{F} := \{(\mathbf{x}, \mathbf{y}, \mathbf{s}) : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{A}^\mathsf{T} \cdot \mathbf{y} - \mathbf{Q} \cdot \mathbf{x} + \mathbf{s} = \mathbf{c}, \mathbf{x} \geq 0, \mathbf{s} \geq 0\}$

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- Strictly feasible set:
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  $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{F}^0$ is a strictly feasible solution, which lies in the *interior* of the region defined by those constraints rather than being on the boundary.

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  $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{F}^0$ is a strictly feasible solution, which lies in the *interior* of the region defined by those constraints rather than being on the boundary. So $\mathcal{F}^0$ is the relative interior of the set $\mathcal{F}$.

Algorithms for IPMs with pure Newton direction

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Exercise 4

Algorithms for IPMs with pure Newton direction

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Consider the quadratic programming problem given below:

Algorithms for IPMs with pure Newton direction

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$$\begin{aligned} \min \quad & x_1 \cdot x_2 + x_1^2 + \frac{3}{2} \cdot x_2^2 + 2 \cdot x_3^2 \\ & +2 \cdot x_1 + x_2 + 3 \cdot x_3 \end{aligned}$$

Algorithms for IPMs with pure Newton direction

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$$
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The current primal-dual estimate of the solution $\mathbf{x^k} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$, $\mathbf{y^k} = (1, \frac{1}{2})^T$, and $\mathbf{s^k} = (\frac{3}{2}, \frac{11}{6}, \frac{10}{3})^T$.

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The current primal-dual estimate of the solution $\mathbf{x^k} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$, $\mathbf{y^k} = (1, \frac{1}{2})^T$, and $\mathbf{s^k} = (\frac{3}{2}, \frac{11}{6}, \frac{10}{3})^T$. Is $(x, y, s) \in \mathcal{F}$?

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The current primal-dual estimate of the solution $\mathbf{x}^\mathbf{k} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$, $\mathbf{y}^\mathbf{k} = (1, \frac{1}{2})^T$, and $\mathbf{s}^\mathbf{k} = (\frac{3}{2}, \frac{11}{6}, \frac{10}{3})^T$. Is $(x, y, s) \in \mathcal{F}$? How about $\mathcal{F}^0$?

Algorithms for IPMs with pure Newton direction

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Two Basic Ingredients of IPMs

Algorithms for IPMs with pure Newton direction

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1. A measure that can be used to evaluate and compare the quality of alternative solutions and search directions.

Algorithms for IPMs with pure Newton direction

Two Basic Ingredients of IPMs

1. A measure that can be used to evaluate and compare the quality of alternative solutions and search directions.
2. A method to generate a better solution, with respect to the measure just mentioned, from a non-optimal solution.

Algorithms for IPMs with pure Newton direction

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Pure Newton Step

Algorithms for IPMs with pure Newton direction

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Assume that we have a current estimate (x^k, y^k, s^k) of the optimal solution to the problem.

Algorithms for IPMs with pure Newton direction

Pure Newton Step

Assume that we have a current estimate (x^k, y^k, s^k) of the optimal solution to the problem.

The Newton step from this point is determined by solving the following system of linear equations:

Algorithms for IPMs with pure Newton direction

Pure Newton Step

Assume that we have a current estimate (x^k, y^k, s^k) of the optimal solution to the problem.

The Newton step from this point is determined by solving the following system of linear equations:

$$\mathbf{J}(\mathbf{x^k}, \mathbf{y^k}, \mathbf{s^k}) \begin{bmatrix} \Delta \mathbf{x^k} \\ \Delta \mathbf{y^k} \\ \Delta \mathbf{s^k} \end{bmatrix} = -\mathbf{F}(\mathbf{x^k}, \mathbf{y^k}, \mathbf{s^k}),$$

Algorithms for IPMs with pure Newton direction

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where $\mathbf{J}(\mathbf{x^k}, \mathbf{y^k}, \mathbf{s^k})$ is the Jacobian of the function F and $[\Delta \mathbf{x^k}, \Delta \mathbf{y^k}, \Delta \mathbf{s^k}]^T$ is the search direction.

Algorithms for IPMs with pure Newton direction

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First, we observe that

Algorithms for IPMs with pure Newton direction

Pure Newton Step

First, we observe that

$$\mathbf{J}(\mathbf{x}^k, \mathbf{y}^k, \mathbf{s}^k) = \begin{bmatrix} -\mathbf{Q} & \mathbf{A}^T & \mathbf{I} \\ \mathbf{A} & 0 & 0 \\ \mathbf{S}^k & 0 & \mathbf{X}^k \end{bmatrix}$$

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where, $\mathbf{X}^k$ and $\mathbf{S}^k$ are diagonal matrices with the components of the vectors $\mathbf{x}^k$ and $\mathbf{s}^k$ along their diagonals.

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So by solving this equation system, we can find the search step for $(k+1)$th iteration.

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Exercise 5

Algorithms for IPMs with pure Newton direction

Pure Newton Step

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Consider the quadratic programming problem given in Exercise 4

Algorithms for IPMs with pure Newton direction

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Consider the quadratic programming problem given in Exercise 4 and the current primal-dual estimate of the solution $\mathbf{x}^\mathbf{k} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$,

Algorithms for IPMs with pure Newton direction

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Algorithms for IPMs with pure Newton direction

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The Newton equation reduces to

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Algorithms for IPMs with pure Newton direction

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Algorithms for IPMs with pure Newton direction

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- To avoid such violations, we sill seek a *step-size parameter* $\alpha^k \in (0, 1]$ such that $\mathbf{x^k} + \alpha^k \cdot \Delta\mathbf{x^k} > 0$ and $\mathbf{s^k} + \alpha^k \cdot \Delta\mathbf{s^k} > 0$.

Algorithms for IPMs with pure Newton direction

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- Once we determine the step-size parameter, we choose the next iterate as

$$(\mathbf{x^{k+1}}, \mathbf{y^{k+1}}, \mathbf{s^{k+1}}) = (\mathbf{x^k}, \mathbf{y^k}, \mathbf{s^k}) + \alpha^k \cdot (\Delta\mathbf{x^k}, \Delta\mathbf{y^k}, \Delta\mathbf{s^k}).$$

Algorithms for IPMs with pure Newton direction

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Weakness of IPMs with pure Newton direction

Algorithms for IPMs with pure Newton direction

Weakness of IPMs with pure Newton direction

- We often can take only a small step along the direction ($\alpha^k \ll 1$) before violating the condition $\mathbf{x^k} + \alpha^k \cdot \Delta\mathbf{x^k} > 0$ and $\mathbf{s^k} + \alpha^k \cdot \Delta\mathbf{s^k} > 0$;

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Algorithms for IPMs with centered Newton direction

Algorithms for IPMs with centered Newton direction

The Central Path

Algorithms for IPMs with centered Newton direction

The Central Path

The central path $\mathcal{C}$ is an arc of strictly feasible points(any point in $\mathcal{C}$ is in $\mathcal{F}^0$) that plays a vital role in the theory of primal-dual algorithm.

Algorithms for IPMs with centered Newton direction

The Central Path

The central path $\mathcal{C}$ is an arc of strictly feasible points(any point in $\mathcal{C}$ is in $\mathcal{F}^0$) that plays a vital role in the theory of primal-dual algorithm. It is parameterized by a scalar $\tau > 0$,

Algorithms for IPMs with centered Newton direction

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The central path $\mathcal{C}$ is an arc of strictly feasible points(any point in $\mathcal{C}$ is in $\mathcal{F}^0$) that plays a vital role in the theory of primal-dual algorithm. It is parameterized by a scalar $\tau > 0$, and the points $(\mathbf{x}_\tau, \mathbf{y}_\tau, \mathbf{s}_\tau)$ on the central path are obtained as solutions of the following system:

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$$(x_\tau)_i \cdot (s_\tau)_i = \tau, \forall i.$$

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- Instead of the complementary condition, we require the products $(x_\tau)_i \cdot (s_\tau)_i$ have the same value for all i.
- The system has a unique solution for every $\tau > 0$, provided that $\mathcal{F}^0$ is nonempty.
- As $\tau \to 0$, the conditions defining the points on the central path approximate the set of optimality conditions more and more closely.

If $\mathcal{F}^0$ is nonempty, $(\mathbf{x}_\tau, \mathbf{y}_\tau, \mathbf{s}_\tau)$ will converge to an optimal solution of the problem.

The Central Path

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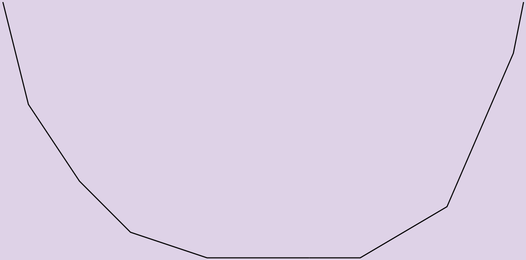

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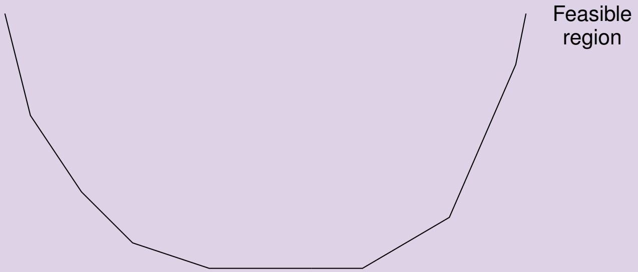

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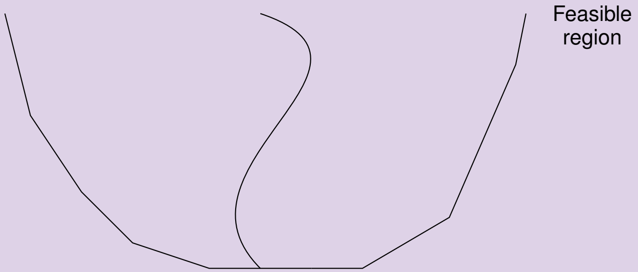

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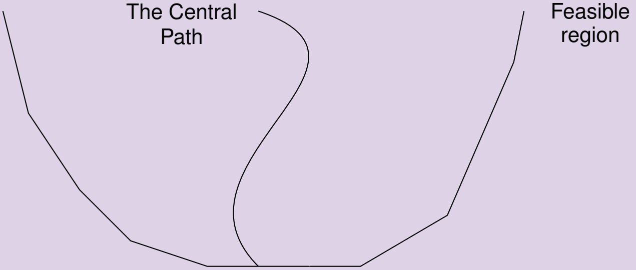

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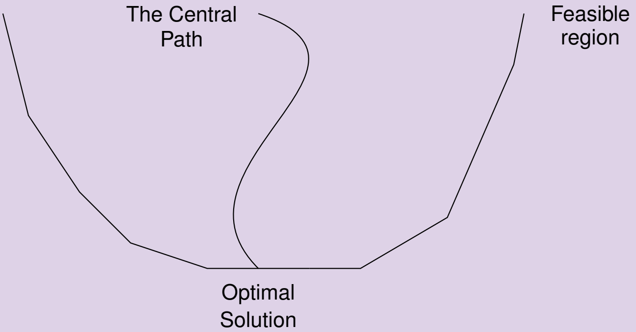

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Exercise 6

The Central Path

Exercise 6

Recall the quadratic programming problem given in Exercise 4

The Central Path

Exercise 6

Recall the quadratic programming problem given in Exercise 4 and the current primal-dual estimate of the solution

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IPMs with Centered Newton directions

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To get over the weakness with pure Newton directions, most interior-point methods take a step toward points on the central path $\mathcal{C}$ corresponding to predetermined value of τ.

IPMs with Centered Newton directions

Centered Newton directions

To get over the weakness with pure Newton directions, most interior-point methods take a step toward points on the central path $\mathcal{C}$ corresponding to predetermined value of τ.

Since such directions are aiming for central points, they are called *centered directions*.

IPMs with Centered Newton directions

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Description

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$$\hat{\mathbf{F}}(\mathbf{x}, \mathbf{y}, \mathbf{s}) = \begin{bmatrix} \mathbf{A}^{\mathsf{T}} \cdot \mathbf{y} - \mathbf{Q} \cdot \mathbf{x} + \mathbf{s} - \mathbf{c} \\ \mathbf{A} \cdot \mathbf{x} - \mathbf{b} \\ \mathbf{X} \cdot \mathbf{S} \cdot \mathbf{e} - \tau \cdot \mathbf{e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

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Since the Jacobian of $\hat{F}$ is identical to the Jacobian of F,

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$$\hat{\mathbf{F}}(\mathbf{x}, \mathbf{y}, \mathbf{s}) = \begin{bmatrix} \mathbf{A}^\mathbf{T} \cdot \mathbf{y} - \mathbf{Q} \cdot \mathbf{x} + \mathbf{s} - \mathbf{c} \\ \mathbf{A} \cdot \mathbf{x} - \mathbf{b} \\ \mathbf{X} \cdot \mathbf{S} \cdot \mathbf{e} - \tau \cdot \mathbf{e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

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$$\mu(\mathbf{x}_\tau, \mathbf{s}_\tau) = \frac{\sum_{i=1}^{n} (x_\tau)_i \cdot (s_\tau)_i}{n} = \frac{\sum_{i=1}^{n} \tau}{n} = \tau.$$

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When $\sigma^k = 1$, we have a *pure centering direction*.

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In most circumstances, the third option is not a good choice as it targets a central point that is "farther" than the current iterate to the optimal solution.

Therefore, we will always choose $\tau \leq \mu(\mathbf{x}, \mathbf{s})$ in defining centered directions.

Generic Interior Point Algorithm

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General Interior Point Algorithm I

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- Choose $(\mathbf{x^0}, \mathbf{y^0}, \mathbf{s^0}) \in \mathcal{F}^0$. For $k = 0, 1, 2, \ldots$ repeat the following steps.

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General Interior Point Algorithm II

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Fortunately, however, we can accommodate infeasible starting points with a small modification of the linear system we solve in each iteration.

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- We only require that the initial point $(\mathbf{x^0}, \mathbf{y^0}, \mathbf{s^0})$ satisfy the nonnegativity restrictions strictly: $\mathbf{x^0} > 0$ and $\mathbf{s^0} > 0$.
- Then the Newton equation from an infeasible point $(\mathbf{x^k}, \mathbf{y^k}, \mathbf{s^k})$ is reduced to

$$\begin{bmatrix} -\mathbf{Q} & \mathbf{A^T} & \mathbf{I} \\ \mathbf{A} & 0 & 0 \\ \mathbf{S^k} & 0 & \mathbf{X^k} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x^k} \\ \Delta\mathbf{y^k} \\ \Delta\mathbf{s^k} \end{bmatrix} = \begin{bmatrix} \mathbf{c} + \mathbf{Q} \cdot \mathbf{x^k} - \mathbf{A^T} \cdot \mathbf{y^k} - \mathbf{s^k} \\ \mathbf{b} - \mathbf{A} \cdot \mathbf{x^k} \\ \tau \cdot \mathbf{e} - \mathbf{X^k} \cdot \mathbf{S^k} \cdot \mathbf{e} \end{bmatrix}$$

  We no longer have zeros in the first and second blocks of the right-hand-side vector since we are not assuming that the iterates satisfy $A \cdot x^k = b$ and $A^T \cdot y^k - Q \cdot x^k + s^k = c$.
- Replacing the linear system in this case, the algorithms can work simultaneously.

Solving Motivating Example 2

Solving Motivating Example 2

Optimization model for Example 2: Quantifying the notion of “risk”

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Solving Motivating Example 2

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Solving Motivating Example 2

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Solving Motivating Example 2

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Solving Motivating Example 2

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Solving Motivating Example 2

Example

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- We wish to invest \$1000.00 in stocks A, B, and C for a one month period.
- We buy a stock at some dollar amount per share in the beginning of the month, and sell it at some dollar amount per share at the end of the month.
- The rate of return of each stock is a random variable with some expected value.
- Our goal is to invest in such a way that the expected end-of-month return is at least \$50.00 or 5%.

Solving Motivating Example 2

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Optimization model for Example 2: The decision variables

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Solving Motivating Example 2

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  Where $\bar{r}_i$ is the expected value of the random variable corresponding to the monthly return per dollar for stock i.

Solving Motivating Example 2

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Optimization model for Example 2

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Solving Motivating Example 2

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Where **x** is the decision vector of size n (n is the number of stocks, $n = 3$ in our example),

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 & \bar{\mathbf{r}}^T \cdot \mathbf{x} & \geq & 50 \\
 & \mathbf{x} & \geq & \mathbf{0}
\end{array}
$$

  Where $\mathbf{x}$ is the decision vector of size n (n is the number of stocks, $n = 3$ in our example), $\mathbf{e}$ is an n-vector of ones, $\bar{\mathbf{r}}$ is the n-vector of expected returns of the stocks, and $\mathbf{Q}$ is the $n \times n$ covariance matrix.

Using MATLAB and Optimization Toolbox Function `quadprog`

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Solving our motivating example using MATLAB

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Using MATLAB and Optimization Toolbox Function `quadprog`

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$$\mathbf{Q} = \begin{bmatrix} 0.0171 & 0.0033 & 0.0012 \\ 0.0033 & 0.0059 & 0.0045 \\ 0.0012 & 0.0045 & 0.0630 \end{bmatrix}$$

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- We find that our optimal solution is $\mathbf{x}^* = \begin{bmatrix} 500 \\ 0 \\ 500 \end{bmatrix}$.

Using MATLAB and Optimization Toolbox Function `quadprog`

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Solving another example using MATLAB

$$\min_{x} \quad \frac{1}{2} \cdot x_1^2 + 3 \cdot x_1 + 4 \cdot x_2$$

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Solving another example using MATLAB

$$\begin{aligned} \min_{x} \quad & \frac{1}{2} \cdot x_1^2 + 3 \cdot x_1 + 4 \cdot x_2 \\ & x_1 + 3 \cdot x_2 \geq 15 \end{aligned}$$

Using MATLAB and Optimization Toolbox Function `quadprog`

Solving another example using MATLAB

$$\min_{x} \quad \frac{1}{2} \cdot x_1^2 + 3 \cdot x_1 + 4 \cdot x_2$$
$$x_1 + 3 \cdot x_2 \quad \geq \quad 15$$
$$2 \cdot x_1 + 5 \cdot x_2 \quad \leq \quad 100$$

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& x_1 + 3 \cdot x_2 \geq 15 \\
& 2 \cdot x_1 + 5 \cdot x_2 \leq 100 \\
& 3 \cdot x_1 + 4 \cdot x_2 \leq 80
\end{aligned}
$$

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& x_1 + 3 \cdot x_2 & \geq & \quad 15 \\
& 2 \cdot x_1 + 5 \cdot x_2 & \leq & \quad 100 \\
& 3 \cdot x_1 + 4 \cdot x_2 & \leq & \quad 80 \\
& x_1, x_2 & \geq & \quad 0
\end{aligned}
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We find that our optimal solution is

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We find that our optimal solution is $\mathbf{x^*} = (0, 5)$.

References

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1. Stephen J. Wright, Primal-Dual Interior-Point Methods, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1997.

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