Robust Optimization: Theory and Tools

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Outline

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   - Introduction
   - Uncertainty sets
   - RO Concepts
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  These uncertainty sets can also arise from statistical estimation in the form of confidence regions.

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The solution must be constraint-robust with respect to the uncertainties of the problem.

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$$\min_{x} f(x)$$

$$G(x,p) \in K, \forall p \in \mathcal{U}$$

Robust feasible set

The robust feasible set is the intersection of the feasible sets $S(p) = \{x : G(x, p) \in K\}$ indexed by the uncertainty set $\mathcal{U}$.

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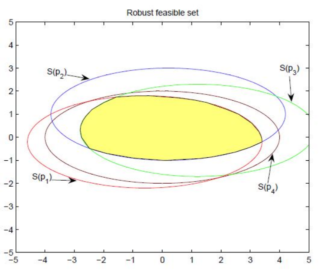

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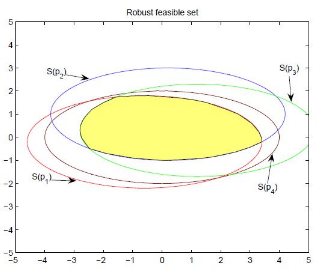

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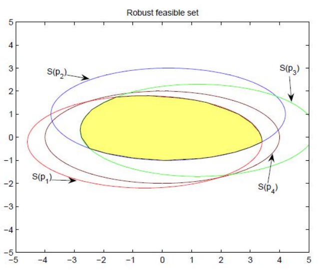

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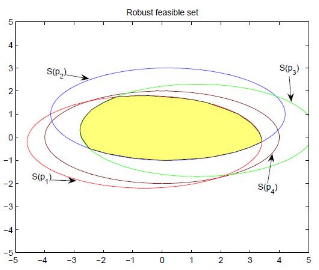

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For an ellipsoidal feasible set with $\mathcal{U} = \{p_1, p_2, p_3, p_4\}$, where p_i correspond to the uncertain center of the ellipse.

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The worst-case behavior of a solution corresponds to the value of the objective function for the worst possible realization of the uncertain data for that particular solution.

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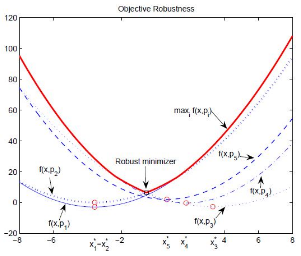

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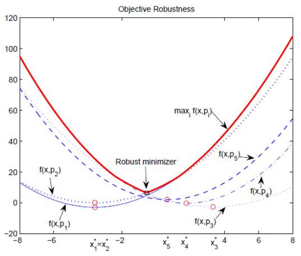

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Outline

1. Theory of Robust Optimization
   - Introduction
   - Uncertainty sets
   - RO Concepts
   - Summary
   - Example
   - Adjustable Robust Optimization (ARO)
2. Tools and Strategies for Robust Optimization
   - Sampling
   - Saddle-Point Characterizations
3. Bibliography

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- Among such solutions, pick one that minimizes the objective function value.

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ARO

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- Assume that A^1 is revealed to the modeler after choosing x^1. So, at the moment of choosing x^2 the modeler knows the value of A^1.
- Could we make robust choices of x^2 taking this sequential nature of the decision process into account?

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- However, ARO is harder to formulate explicitly as an easy optimization problem e.g., linear programs.
- Because we do not know the functional form of the dependency.
- Only very simple uncertainty sets allow “nice” ARCs. Otherwise, we have to assume a simple functional form for dependencies, e.g., affine dependency.

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$$f(x^*, p) \leq f(x^*, p^*) \leq f(x, p^*),\ \forall x \in S, p \in \mathcal{U}$$

Software

Robust Optimization and Robust Programming software:

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http://www.aimms.com/operations-research/mathematical-programming/robust-optimization/

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Conclusion

Robustness* $=$ *best solution against the worst possible data realization.

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