Stochastic Programming: Theory and Algorithms

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Introduction

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- For example, to represent the outcomes of flipping a fair coin twice in a row, we would use four random events $\Omega = \{HH, HT, TH, TT\}$, each with probability $1/4$.

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- Stochastic programming models that include both anticipative and adaptive variables are called recourse models.

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- These problems can be formulated conveniently as multi-stage stochastic programming problems with recourse.

Probabilistic Constraints

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Now:

- For each pair of values (a_1, a_2) we have an associated joint probability $(1/36)$
- Given values for $x \geq 0$ and $y \geq 0$ we can easily check whether the constraint is true with probability $1 - \alpha$.

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- In this case, $x = 0$ and $y = 1$ is not a feasible solution, since we already have a probability of 2/36 = 0.0555 that the constraint is infeasible.
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  It is impossible for the constraint to be feasible with probability 0.95 (since 1-0.0555 = 0.9445).
- This particular problem (because it contains just two variables) can be easily solved by a simple numeric search procedure.
- For example, for $\alpha = 0.01$ the solution is $x = 3$, $y = 0$ and for $\alpha = 0.05$ the solution is $x = 1$, $y = 1$

Recourse Problems

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- Notice that the first-stage decisions will not necessarily satisfy the linking constraints $B(\omega)x + C(\omega)y(\omega) = d(\omega)$,
- Therefore, recourse allows one to make sure that the initial decisions can be ”corrected” with respect to this second set of feasibility equations.

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$$\begin{aligned} f(x, \omega) &= \max c(\omega)^T y(\omega) \qquad (2) \\ C(\omega)y(\omega) &= d(\omega) - B(\omega)x \\ y(\omega) &\geq 0 \end{aligned}$$

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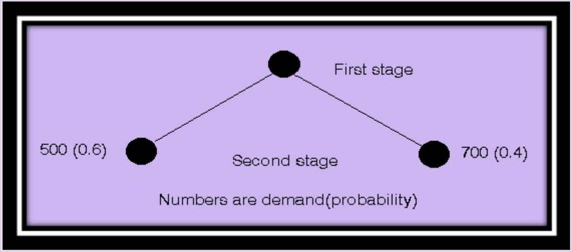

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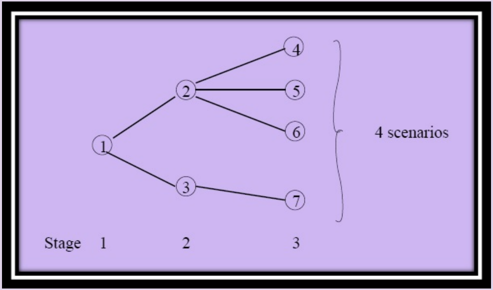

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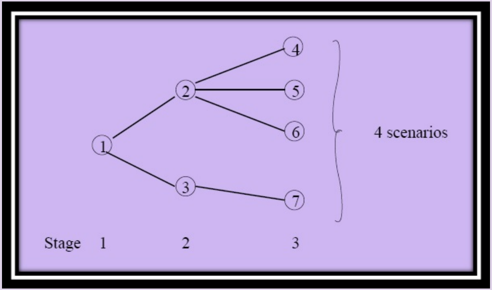
  - Each node is in one stage.
  - The root is the unique node in stage 1.
  - Each node i in stage $k \geq 2$ is adjacent to a unique node $a(i)$ in stage $k-1$.
  - Node $a(i)$ is called the father of node i.
- The paths from the root to the leaves (in stage n) represent the scenarios.
  - Thus the last stage has as many nodes as scenarios.
  - These nodes are called the terminal nodes.
  - The collection of scenarios passing through node i in stage k have identical components $(o_1, \ldots, o_{k-1})$.

Multi-Stage Problems



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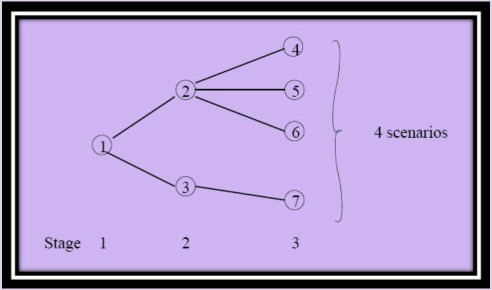

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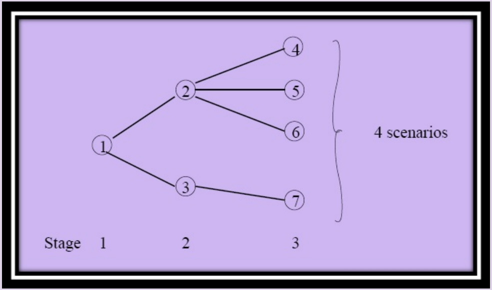

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- The father of Node 6 is Node 2, in other words $a(6) = 2$.

Multi-Stage Problems

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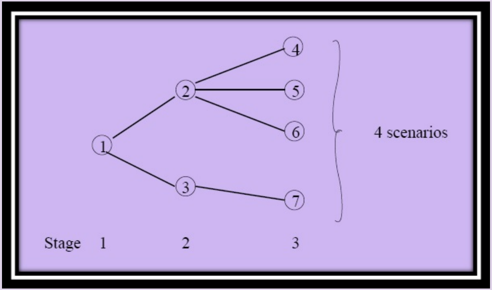
Multi-Stage Problems - Exercise 3

Exercise 3: Develop the Linear program (6) for the following scenario tree:



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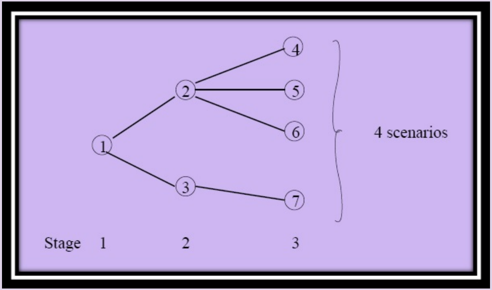

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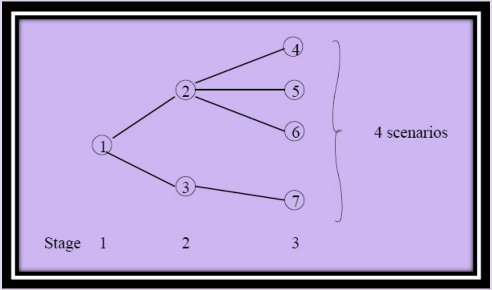

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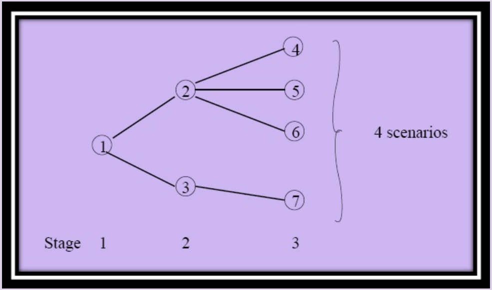

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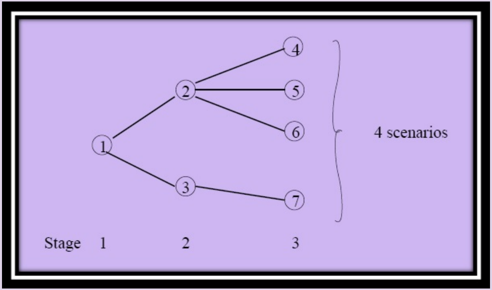

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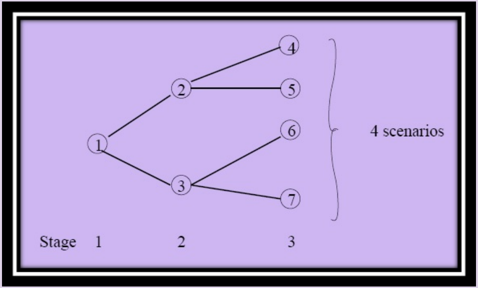

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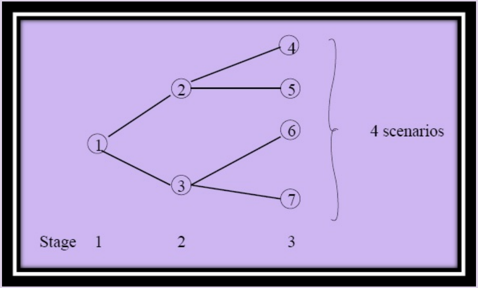

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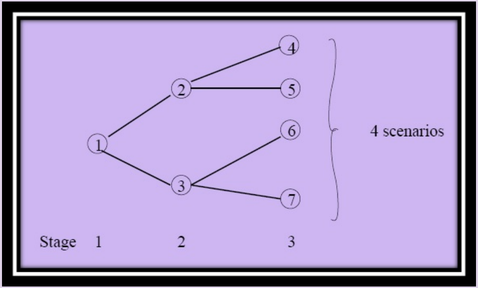

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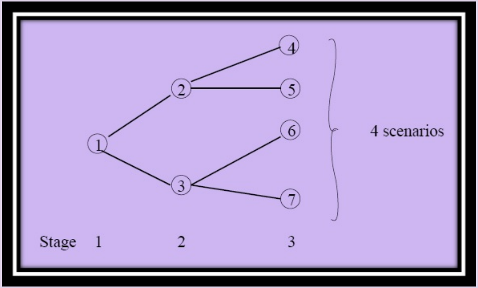

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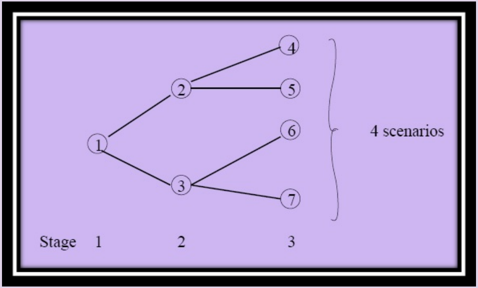

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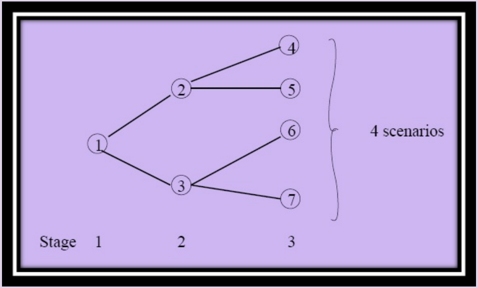

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Multi-Stage Problems - Exercise 5

Exercise 5: Extend the two-stage problem we have considered in Exercise 2 to three stages.

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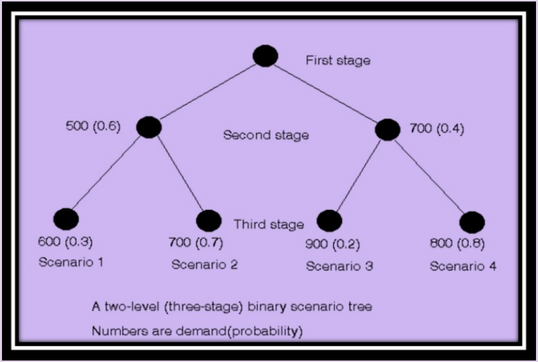

Multi-Stage Problems - Exercise 5

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  - a demand of 500 with probability 0.6

Multi-Stage Problems - Exercise 5

- We initially make a decision about how much to produce.
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  - a demand of 500 with probability 0.6
  - a demand of 700 with probability 0.4

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- We initially make a decision about how much to produce.
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  - a demand of 500 with probability 0.6
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- After this realization, we make a decision as to how much to produce to meet demand in the next period (THE THIRD-STAGE).

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- For example, if the realized demand at the second stage was 500, then the possible realizations at the third stage are:
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- Note here at each level in the scenario tree the appropriate probabilities must sum to one.

Multi-Stage Problems - Exercise 5

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Multi-Stage Problems - Exercise 5

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- Note here that these probabilities add to one (these 4 scenarios are the only possible futures that we can have).

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Multi-Stage Problems - Exercise 5

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Multi-Stage Problems - Exercise 5

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Multi-Stage Problems - Exercise 5

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2	0.42	$2 \cdot x_{22} + 3 \cdot y_{22} + 3 \cdot y_{32}$

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Multi-Stage Problems - Exercise 5

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&+ 0.08 \cdot (2 \cdot x_{23} + 3 \cdot y_{23} + 3 \cdot y_{33}) + 0.32 \cdot (2 \cdot x_{24} + 3 \cdot y_{24} + 3 \cdot y_{34})
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$$x_1 + y_{2s} - 500 + x_{2s} + y_{3s} \geq 600 \qquad (s = 1)$$

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subject to

$$\begin{aligned} x_1 + y_{2s} - 500 + x_{2s} + y_{3s} &\geq 600 \qquad (s = 1) \\ x_1 + y_{2s} - 500 + x_{2s} + y_{3s} &\geq 700 \qquad (s = 2) \end{aligned}$$

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$$x_1 + y_{2s} - 500 + x_{2s} + y_{3s} \geq 600 \qquad (s = 1)$$
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$$x_1 + y_{2s} - 700 + x_{2s} + y_{3s} \geq 900 \qquad (s = 3)$$
$$x_1 + y_{2s} - 700 + x_{2s} + y_{3s} \geq 800 \qquad (s = 4)$$
$$y_{21} = y_{22}, \quad x_{21} = x_{22}, \quad y_{23} = y_{24}, \quad x_{23} = x_{24}$$
$$all \;\; variables \geq 0$$

Multi-Stage Problems - Exercise 5

Solving the previous problem using LP solver:

Multi-Stage Problems - Exercise 5

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- Cost will be 2664

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Multi-Stage Problems - Exercise 5

Solving the previous problem using LP solver:

- Cost will be 2664
- $x_1 = 700$
- $y_{21} = y_{22} = 0, \quad y_{23} = y_{24} = 0$

Multi-Stage Problems - Exercise 5

Solving the previous problem using LP solver:

- Cost will be 2664
- $x_1 = 700$
- $y_{21} = y_{22} = 0, \quad y_{23} = y_{24} = 0$
- $x_{21} = x_{22} = 500$

Multi-Stage Problems - Exercise 5

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Multi-Stage Problems - Exercise 5

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- $x_1 = 700$
- $y_{21} = y_{22} = 0, \quad y_{23} = y_{24} = 0$
- $x_{21} = x_{22} = 500$
- $y_{31} = y_{32} = 0$
- $x_{23} = x_{24} = 800$
- $y_{33} = 100, y_{34} = 0$

Decomposition

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- The size of the linear program (6) depends on the number of decision stages and the branching factor at each node of the scenario tree.

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- The size of the linear program (6) depends on the number of decision stages and the branching factor at each node of the scenario tree.
- For example, a 4-stage model with 25 branches at each node has $25x25x25x25 = 390625$ scenarios.

Decomposition

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- Obviously, the size of (6) can be a limiting factor in solving realistic problems.
- When this occurs, it becomes essential to take advantage of the special structure of the linear program (6).

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- The master problem and recourse problems are linear programs.
- The size of these linear programs is much smaller than the size of full model (5).

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As before, upper and lower bounds are computed at each iteration and the algorithm stops when the difference drops below a given tolerance.

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  - Random Sampling and Tree Fitting are used.
- On the other hand, the linear program (6) can only be solved if the size of the scenario tree is reasonably small, suggesting a rather limited number of scenarios.

References

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