Linear Programming - Duality and Applications

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Outline

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Review of Concepts

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1. Convex sets and convex functions.

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6. Finding and initial basis.

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1. Convex sets and convex functions.
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4. Main ideas of the Simplex method.
5. Local optimality conditions for the Simplex method.
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7. Other methodologies.

Convex sets and Convex functions

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A set S is said to be convex, if:

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Definition (Convex Combination)

Given two points $\mathbf{x}$ and $\mathbf{y}$ in E^m, and $\alpha \in [0,1]$, the parametric point $\alpha \cdot \mathbf{x} + (1-\alpha) \cdot \mathbf{y}$ is said to be a convex combination of $\mathbf{x}$ and $\mathbf{y}$.

Local optimum and Global optimum

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Consider the following optimization problem:

$$\begin{aligned} & \min_{\mathbf{x}} f(x) \\ s.t. \quad & \mathbf{x} \in \mathbf{S} \end{aligned}$$

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$$\min_{\mathbf{x}} f(x)$$
$$s.t. \quad \mathbf{x} \in \mathbf{S}$$

If S is a convex set and f is a convex function of $\mathbf{x}$ *on S, the all local optima are also global optima.*

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2. Partition **A** into (**B** : **N**) and **c** into ($\mathbf{c_B}$: $\mathbf{c_N}$), such that **B** is a feasible basis.
3. If **B** is not an optimal basis, exchange a column of **B** with a column of **N**, such that the basis structure is preserved and there is finite improvement in the solution.

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   The canonical form of z and $\mathbf{x_B}$ can be written as:

$$z = \mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{b} - \sum_{j \in J} (\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{a}_j - c_j) x_j$$

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   where J denotes the index set of the nonbasic variables.

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$$\frac{\partial z}{\partial x_j} = -(z_j - c_j) = -(\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{a}_j - c_j) \leq 0, \text{ for all } j \in J$$

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or, equivalently, if $(z_j - c_j) = (\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{a}_j - c_j) \geq 0$, for all $j \in J$.

Finding an initial basis

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Consider the system:

$$\max 3 \cdot x_1 - 4 \cdot x_2 \quad \mathbf{x} \geq \mathbf{0}$$

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

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Change the system to:

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Finally drive x_5 and x_6 out of the system, by changing the system to:

$$\max -x_5 - x_6 \quad \mathbf{x} \geq \mathbf{0}$$

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Other Methodologies

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A Motivating Example

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Establishing bounds on z^*

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Consider the point $(0, 0, 1, 0)$. Can you conclude $z^* \geq 5$.

From the point $(3, 0, 2, 0)$, we can conclude that $z^* \geq 22$.

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Consider the point $(0, 0, 1, 0)$. Can you conclude $z^* \geq 5$.

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How about an upper bound? (3)+(4) gives $4 \cdot x_1 + 3 \cdot x_2 + 6 \cdot x_3 + 3 \cdot x_4 \leq 58$.

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Can you conclude $z^* \leq 58$?

Finding bounds

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Establishing an upper bound

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Finding bounds

Establishing an upper bound

In general, you want the linear combination of constraints that provides the smallest upper bound.

How to find this linear combination?

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Dual of the Canonical form

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$$z = \max \mathbf{c} \cdot \mathbf{x}$$

$$\begin{array}{rcl} \mathbf{A} \cdot \mathbf{x} & \leq & \mathbf{b} \\ \mathbf{x} & \geq & 0 \end{array}$$

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Duals exist for general forms of linear programs as well.

Example

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Find the dual of:

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$$
\begin{aligned}
\max 4 \cdot x_1 + 2 \cdot x_2 & & \\
x_1 + x_2 & \leq & 2 \\
x_1 + 2 \cdot x_2 & \leq & 15 \\
2 \cdot x_1 - x_2 & \leq & 12 \\
x_1, x_2 & \geq & 0
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The dual of the dual is the primal.

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The Weak Duality theorem

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where $\mathbf{x}'$ *and* $\mathbf{y}'$ *are any primal feasible and dual feasible solution respectively.*

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Since $\mathbf{x}'$ is primal feasible, we must have,

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It follows that $\mathbf{y}' \cdot \mathbf{A} \cdot \mathbf{x}' \leq \mathbf{y}' \cdot \mathbf{b}$ and $\mathbf{y}' \cdot \mathbf{A} \cdot \mathbf{x}' \geq \mathbf{c} \cdot \mathbf{x}'$.

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The theorem follows.

Consequences of the weak duality theorem

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What is the primal dual relationship in the following linear program:

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What is the primal dual relationship in the following linear program:

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-x_1 + 2 \cdot x_2 & \leq & -2 \\
x_1 - 2 \cdot x_2 & \leq & -2 \\
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Optimality theorem from Weak duality

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Theorem

Optimality theorem from Weak duality

Theorem

If $\mathbf{x}$ is primal feasible and $\mathbf{y}$ is dual feasible, and $\mathbf{c} \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{b}$, then $\mathbf{x}$ is primal optimal and $\mathbf{y}$ is dual optimal.

The Strong Duality Theorem

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As per the weak duality theorem, the feasibility of the primal implies a finite optimal for the dual and the feasibility of the dual implies a finite optimal for the primal.

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Then the optimal point is $\mathbf{x} =$

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Let $\mathbf{B}$ denote the optimal basis of the primal in standard form.

Then the optimal point is $\mathbf{x} = \binom{\mathbf{B}^{-1} \cdot \mathbf{b}}{\mathbf{0}}$ and the the optimal solution for the primal is $z = \mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{b}$.

Proof of strong duality (contd.)

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What we need now is a feasible dual having the same solution value as z.

Proof of strong duality (contd.)

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The value of the dual at this point is:

Proof of strong duality (contd.)

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What we need now is a feasible dual having the same solution value as z.

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Since $\mathbf{B}$ is an optimal basis, we must have $(z_j - c_j) \geq 0$ for all the columns of $(\mathbf{A}, \mathbf{I})$.

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It follows that $\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{A} - \mathbf{c} \geq 0$ and $\mathbf{c_B} \cdot \mathbf{B}^{-1} \cdot \mathbf{I} \geq \mathbf{0}$.

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What we need now is a feasible dual having the same solution value as z.

Consider $\mathbf{y} = \mathbf{c_B} \cdot \mathbf{B^{-1}}$.

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It follows that $\mathbf{c_B} \cdot \mathbf{B^{-1}} \cdot \mathbf{A} - \mathbf{c} \geq 0$ and $\mathbf{c_B} \cdot \mathbf{B^{-1}} \cdot \mathbf{I} \geq \mathbf{0}$.

In other words, $\mathbf{c_B} \cdot \mathbf{B^{-1}} \cdot \mathbf{A} \geq \mathbf{c}$ and $\mathbf{c_B} \cdot \mathbf{B^{-1}} \geq \mathbf{0}$.

Complementary Slackness

Complementary Slackness

Theorem

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Theorem

Let $\mathbf{s} = \mathbf{b} - \mathbf{A} \cdot \mathbf{x}$ *denote the set of slack variables and let* $\mathbf{t} = \mathbf{y} \cdot \mathbf{A} - \mathbf{c}$ *denote the vector of surplus variables.*

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Observe that,

$$\mathbf{c} \cdot \mathbf{x}^* \quad =$$

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Complementary Slackness

Theorem

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The theorem follows.

Application of concepts

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Example

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Solve the linear program

Application of concepts

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Solve the linear program

$$\max 10 \cdot x_1 + 6 \cdot x_2 - 4 \cdot x_3 + x_4 + 12 \cdot x_5$$
$$2 \cdot x_1 + x_2 + x_3 + 3 \cdot x_5 \leq 18$$
$$x_1 + x_2 - x_3 + x_4 + 2 \cdot x_5 \leq 6$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Geometric interpretation of Duality

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Karush Kuhn Tucker conditions

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Theorem

At the primal optimal solution, the gradient of the objective function can be written as a non-negative linear combination of the gradients of the binding *constraints.*

Applications to Finance

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Applications

1. Short term financing.

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1. Short term financing.
2. Dedication.

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1. Short term financing.
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3. Arbitrage.

Applications to Finance

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1. Short term financing.
2. Dedication.
3. Arbitrage.
4. Derivative securities and asset pricing.

Short-term financing

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7. Any paper issued in January through March requires a 2% interest rate payment three months later.

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Objective Function

Modeling

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Objective Function

$$\max v$$

Modeling (contd.)

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Constraints

$$
\begin{aligned}
x_1 + y_1 - z_1 &= 150 \\
x_2 + y_2 - 1.01 \cdot x_1 + 1.003 \cdot z_1 - z_2 &= 100 \\
x_3 + y_3 - 1.01 \cdot x_2 + 1.003 \cdot z_2 - z_3 &= -200 \\
x_4 - 1.02 \cdot y_1 - 1.01 \cdot x_3 + 1.003 \cdot z_3 - z_4 &= 200 \\
x_5 - 1.02 \cdot y_2 - 1.01 \cdot x_4 + 1.003 \cdot z_4 - z_5 &= -50 \\
-1.02 \cdot y_3 - 1.01 \cdot x_5 + 1.003 \cdot z_5 - v &= -300 \\
x_i &\leq 100, \ \ i = 1, 2, 3, 4, 5 \\
x_i, y_i, z_i &\geq 0
\end{aligned}
$$

Dedication (Cash flow matching)

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1. Technique to fund known liabilities in the future.
2. Form a portfolio of assets, whose cash inflows exactly offset the cash outflows of liabilities.
3. The liabilities will thus be paid off without the need to buy or sell future assets.
4. Typically, such a portfolio consists of risk-free bonds.

Arbitrage

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1. has a positive initial cash flow and has no risk of a loss later (type A), or
2. requires no initial cash input, has no risk of loss and a positive probability of making profits in the future (type B).

Arbitrage (contd.)

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Let S^i, $i = 1, 2, \ldots, n$ denote a collection of securities.
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Let $S_1^i(\omega_j)$ denote the price of security S^i at time 1 and state ω_j.

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We assume that $S_0^0 = 1$ and $S_1^0(\omega_j) = R = 1 + r$, $\forall j$.

Arbitrage (contd.)

Definition

Let S^i, $i = 1, 2, \ldots, n$ denote a collection of securities.
Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ denote m distinct states.
Let $S^i_1(\omega_j)$ denote the price of security S^i at time 1 and state ω_j.
We use S^0 to denote the riskless security that pays interest r% at time 1.
We assume that $S^0_0 = 1$ and $S^0_1(\omega_j) = R = 1 + r, \forall j$.
A risk-neutral probability measure on Ω is a positive vector $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ such that $\sum_{j=1}^m p_j = 1$ and for every security S^i, $i = 0, 1, \ldots, n$,

$$S^i_0 = \frac{1}{R}\left(\sum_{j=1}^{m} p_j \cdot S^i_1(\omega_j)\right) = \frac{1}{R}\mathbf{E}[S^i_1],$$

where $\mathbf{E}[S]$ denotes the expected value of the random variable S, under the probability distribution $\mathbf{p}$.

Asset Pricing

Asset Pricing

Theorem

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Theorem

A risk neutral probability measure exists if and only if there is no arbitrage.