Linear Programming - Theory and Algorithms

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Outline

1. Motivating Examples

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3. The return for a unit of β is \$30.
4. Each unit of β requires 3 hours of assembly and 2 hours of testing.

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6. We have a total of 240 hours available for assembly and 140 hours for testing.

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How many units of α and β should be produced to maximize our return?

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How should we invest the money in order to maximize our return?

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How many acres of each product should be grown to maximize our profit?

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	Client			
Warehouse	1	2	3	4
1	7	3	8	4
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How can we perform the shipping while minimizing our shipping cost?

Basic Steps

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1. Determine the decision (or control or structural) variables.

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General Form of a Linear Program

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$$x_1, \ldots, x_n \geq 0$$

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4. Divisibility - Variables are continuous and not discrete.

Forms of a linear program

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Converting linear programs into standard form

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Variables

If a variable (say x_1) is unrestricted in sign, replace it with $x_1' - x_1''$, where both $x_1', x_1'' \geq 0$.

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Both slack and surplus variables are inherently non-negative.

Equivalence of the feasibility and optimization versions

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Convert the following Linear Program into Standard Form.

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subject to

$$\begin{aligned} -x_1 + 3 \cdot x_2 - x_3 + 2 \cdot x_4 &\leq -12 \\ 5 \cdot x_1 + x_2 + 4 \cdot x_3 - x_4 &\geq 10 \\ 3 \cdot x_1 - 2 \cdot x_2 + x_3 - x_4 &= -8 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Solution

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$$-x_1 + 3 \cdot x_2 - x_3 + 2 \cdot x_4 + s_1 = -12$$
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Adding the bounds on the slack and surplus variables

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Finally, converting the objective function

$$\text{Maximize } z = -2 \cdot x_1 + 3 \cdot x_2 - 5 \cdot x_3 - x_4 + 0s_1 + 0s_2$$

Representing constraints as sections of a plane

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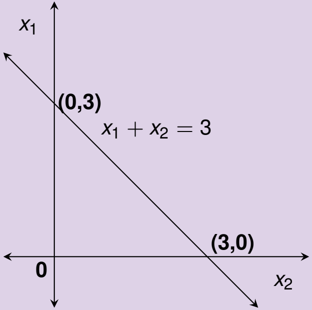
Geometric View of Constraints

Representing constraints as sections of a plane

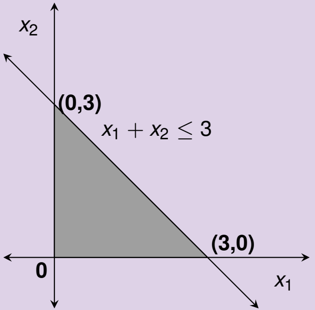
Geometric View of Constraints

An equality, such as $x_1 + x_2 = 3$, can be viewed as a line in the x_1,x_2 plane.



Geometric View of Constraints

Similarly an inequality, such as $x_1 + x_2 \leq 3$, can be viewed as as the half plane above or below a line in the x_1,x_2 plane.



Geometric View of Constraints

For a system of constraints the section of the plane corresponding to solutions to that system is simply the intersection of the portions of the plane corresponding to each constraint. For instance, the constraints

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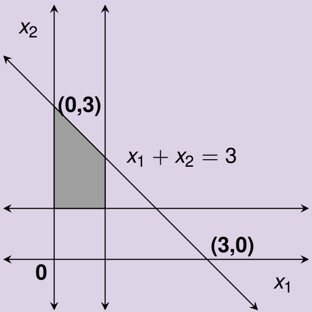

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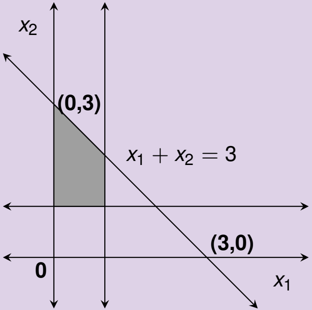
It also helps to find the gradient of z as it identifies the direction in which z grows the fastest.
As the objective function is of the form $z = c_1 \cdot x_1 + c_2 \cdot x_2$, the gradient is simply the vector (c_1, c_2).

Handling the objective function

For example adding the objective function $z = x_1 + 2x_2$ to our previous example yields



Thus trying to maximize z would yield that maximum z to be 6 when $x_1 = 0$ and $x_2 = 3$. Similarly trying to minimize z would yield that minimum z to be 2 when $x_1 = 0$ and $x_2 = 1$.

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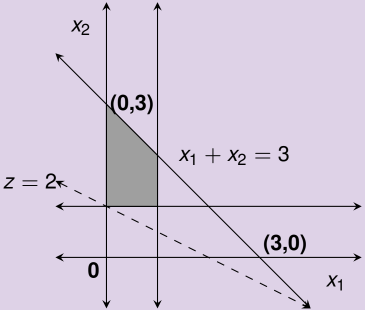

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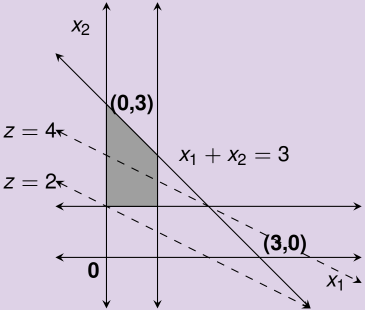

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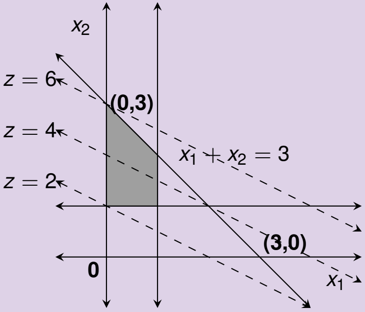

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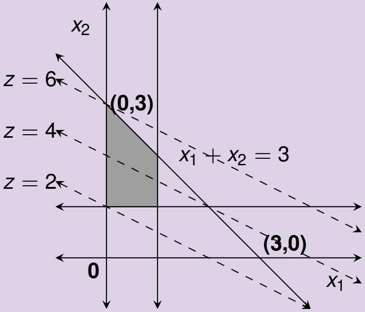

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Exercise 1

Solve the following linear program graphically

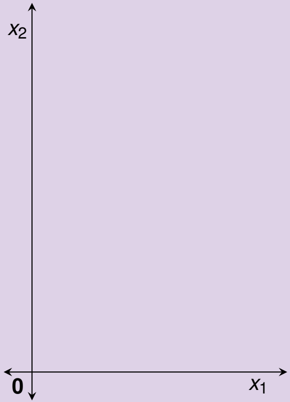
$$\text{minimize } z = 4 \cdot x_1 + 5 \cdot x_2$$

subject to

$$\begin{aligned} 3 \cdot x_1 + 2 \cdot x_2 &\leq 24 \\ x_1 &\geq 5 \\ 3 \cdot x_1 - x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution

If the constraints are plotted onto a graph we see



There are no points which satisfy all three constraints. Thus no solution exists.

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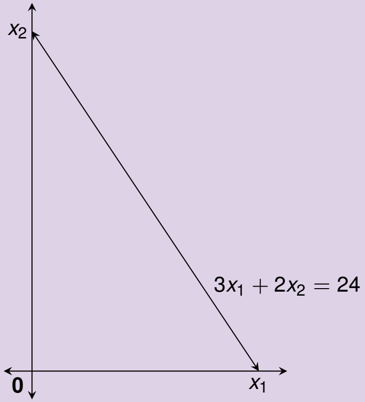

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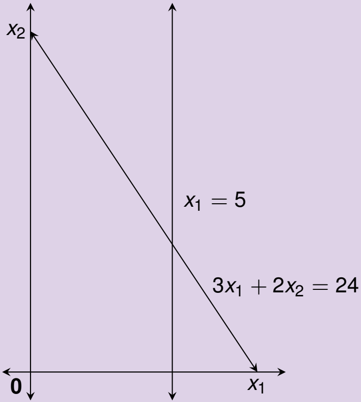

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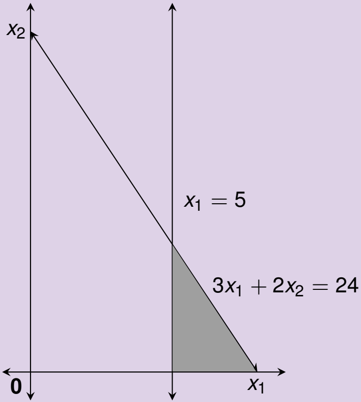

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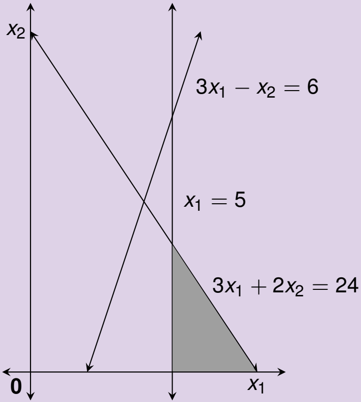

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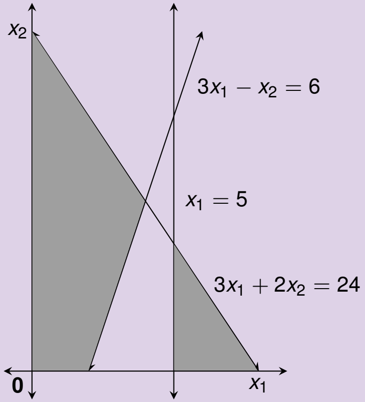

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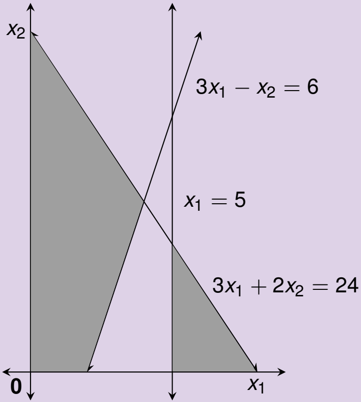

There are no points which satisfy all three constraints. Thus no solution exists.

Exercise 2

Solve the following system of constraints graphically

$$\text{minimize } z = x_1 - 4 \cdot x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 12 \\ -2 \cdot x_1 + x_2 &\leq 4 \\ x_2 &\leq 8 \\ x_1 - 3 \cdot x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution

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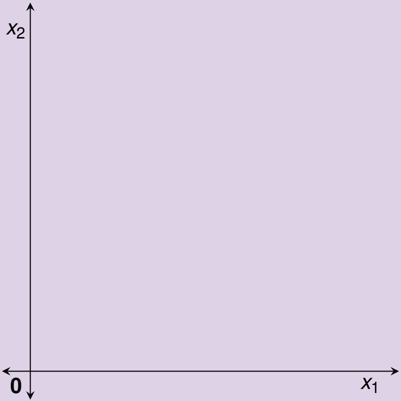
Plotting the constraints and then checking various values of z we get.



Thus the minimum value of z is $z = -30$ and occurs at $(x_1, x_2) = (2, 8)$.

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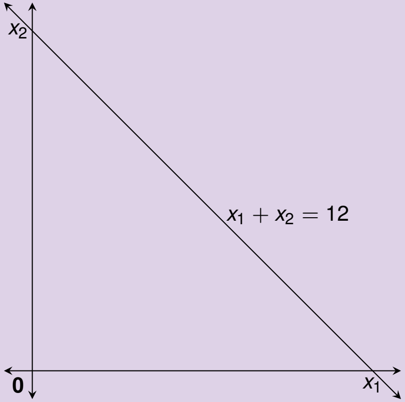

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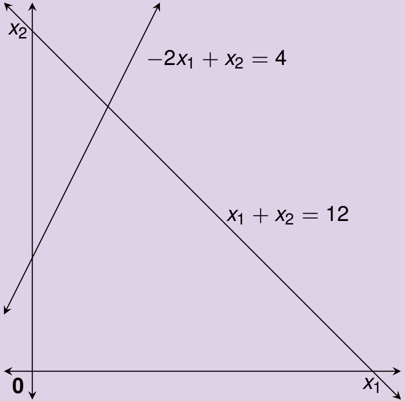

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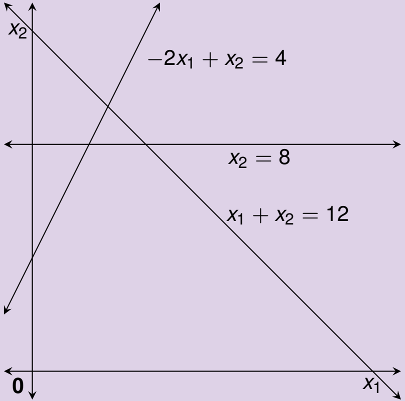

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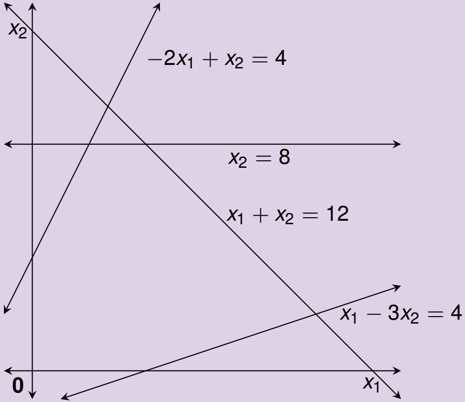

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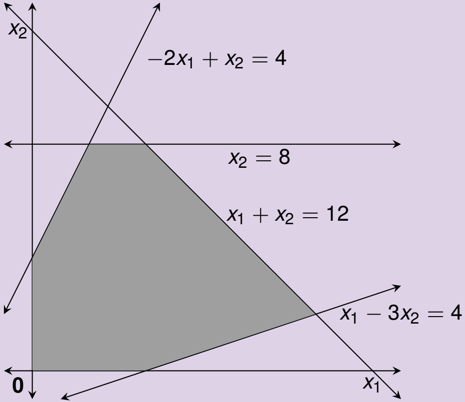

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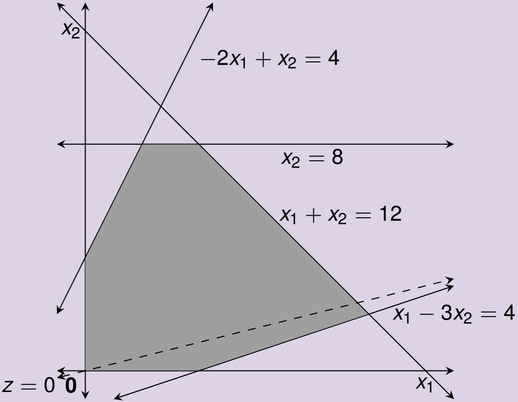

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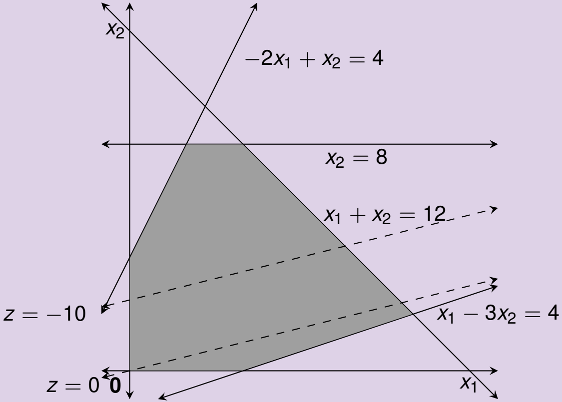

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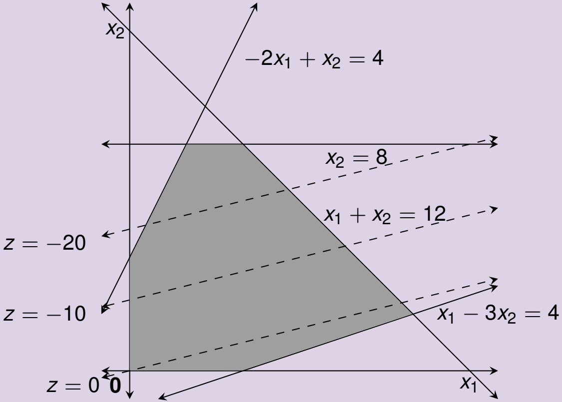

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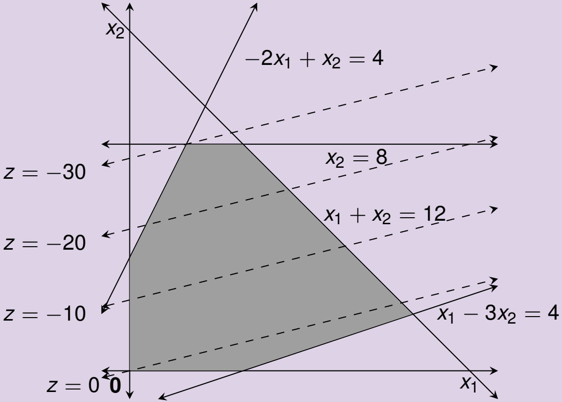

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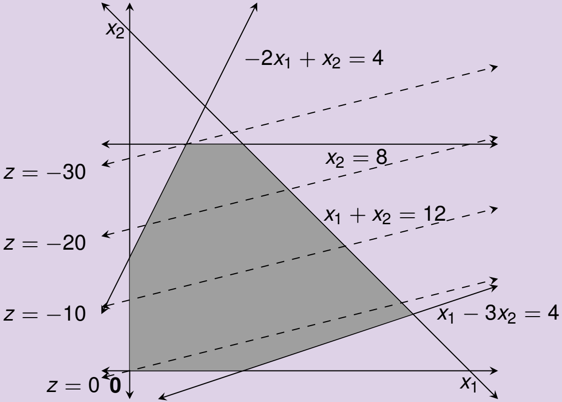

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Exercise 3

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Solve the following linear program graphically

$$\text{maximize } z = x_1 + 2 \cdot x_2$$

subject to

$$\begin{aligned} -2 \cdot x_1 + x_2 &\leq 2 \\ 2 \cdot x_1 + 5 \cdot x_2 &\geq 10 \\ x_1 - 4 \cdot x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution

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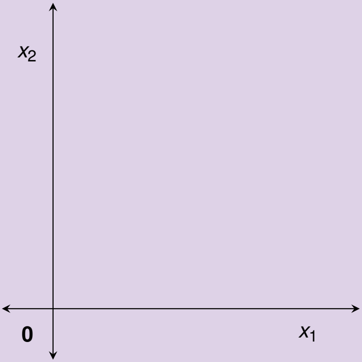

Thus there is no maximum value of z as z can be increased indefinitely and the system will still be feasible.

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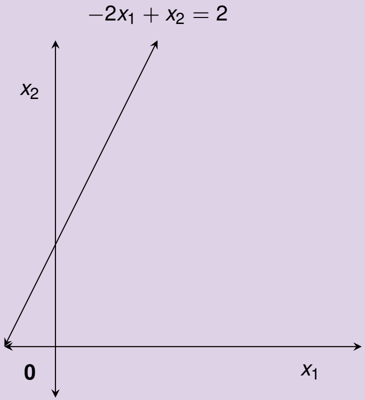

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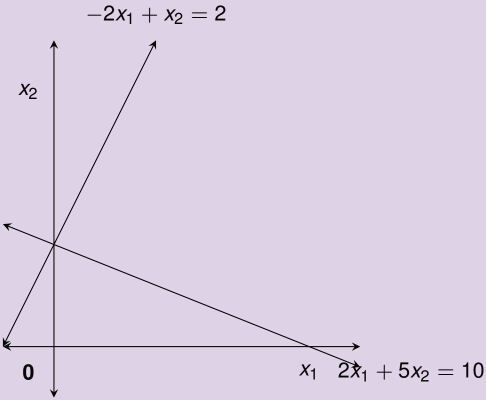

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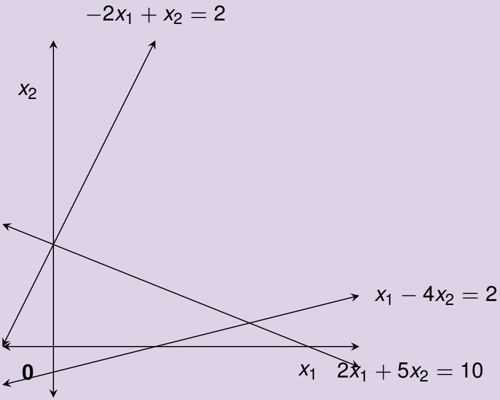

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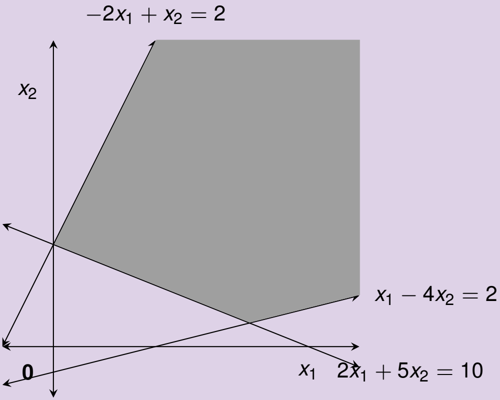

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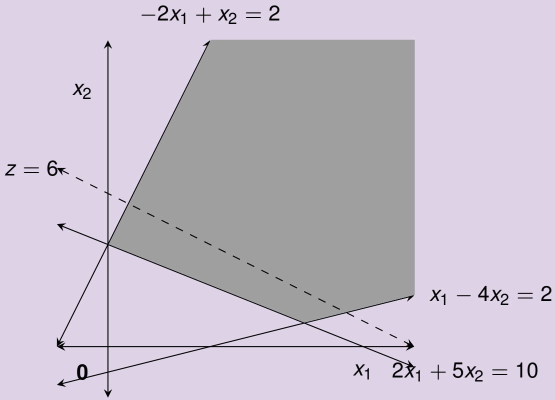

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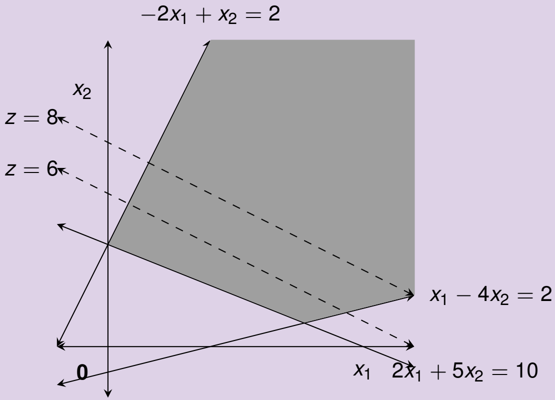

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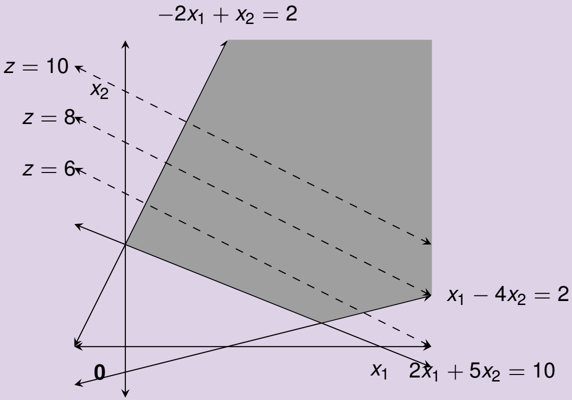

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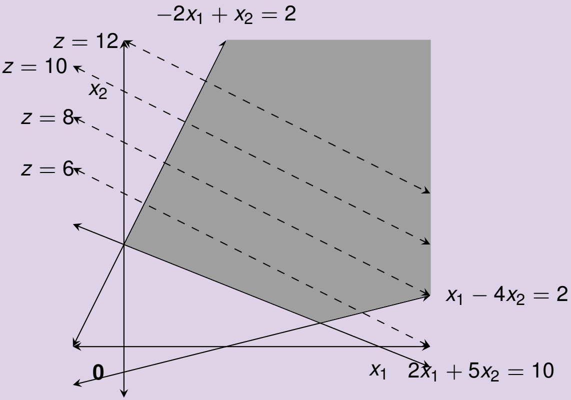

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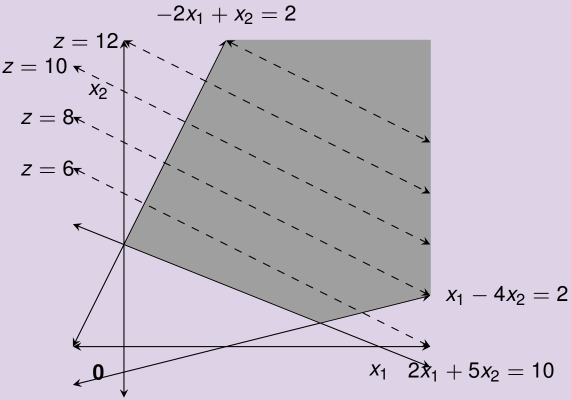

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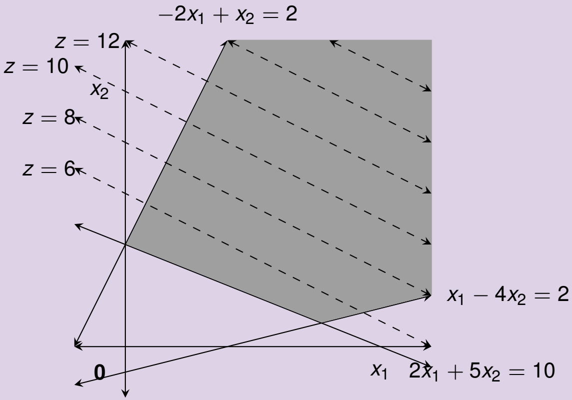

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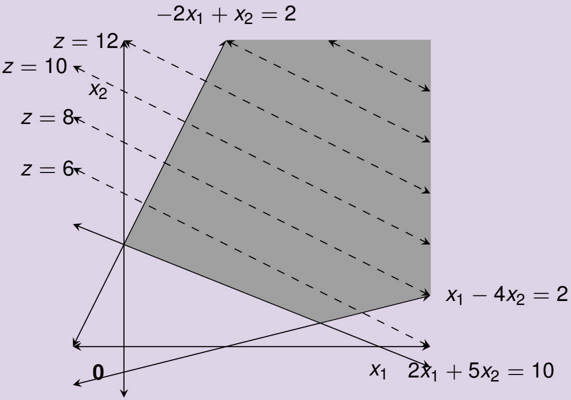

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Hyperplanes and Halfspaces

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A hyperplane is a set of points, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$, that satisfy $\mathbf{a} \cdot \mathbf{x} = b$, where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and b is a scalar.

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A closed halfspace corresponding to a hyperplane $\mathbf{ax} = b$ is either of the sets $H^+ = \{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} \geq b\}$ or $H^- = \{\mathbf{x} : \mathbf{a} \cdot \mathbf{x} \leq b\}$.

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A set, S, is convex if for any two points $\mathbf{x}_1, \mathbf{x}_2 \in S$ then all points on the line segment connecting $\mathbf{x}_1$ and $\mathbf{x}_2$ are in S. This means that $\forall \alpha \in [0, 1]$, $\alpha \mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2 \in S$.

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Systems of constraints as Polyhedral Sets

A constraint system of the form $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is a polyhedral set as each constraint corresponds to a halfspace.

Convexity of polyhedra

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Theorem

The set $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *is convex.*

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Mathematically there do not exist $\mathbf{x}_1, \mathbf{x}_2 \in S$, $\mathbf{x}_1 \neq \mathbf{x}_2$, and $\alpha \in (0, 1)$ such that $\mathbf{x} = \alpha \cdot \mathbf{x}_1 + (1 - \alpha) \cdot \mathbf{x}_2$.

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Let $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, *where* $\mathbf{A}$ *is* $m \times n$ *and* $\mathit{rank}(\mathbf{A}) = m < n$.

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We will only be dealing with bounded polyhedra for the rest of this topic.

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1. S *has at least one extreme point and at most a finite number of extreme points, thus* $E = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \neq \emptyset$.
2. *if* $\mathbf{x} \in S$*, then* $\mathbf{x}$ *can be written as a convex combination of extreme points*

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Let $S = \{\mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ *and consider the following linear program.*

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We will refer to $\mathbf{B}$ as the *basis matrix*.

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We can rewrite $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ as $\mathbf{B} \cdot \mathbf{x_B} + \mathbf{N} \cdot \mathbf{x_N} = \mathbf{b}$, where $\mathbf{x} = \binom{\mathbf{x_B}}{\mathbf{x_N}}$.

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Thus, $\mathbf{B}^{-1} \cdot \mathbf{B} \cdot \mathbf{x_B} + \mathbf{B}^{-1} \cdot \mathbf{N} \cdot \mathbf{x_N} = \mathbf{B}^{-1} \cdot \mathbf{b}$.

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If we set $\mathbf{x_N} = \mathbf{0}$ then $\mathbf{x_B} = \mathbf{B}^{-1} \cdot \mathbf{b}$ and $\mathbf{x} = \binom{\mathbf{B}^{-1} \cdot \mathbf{b}}{\mathbf{0}}$.

This value is called a *basic solution*.

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Connecting extreme points and basic feasible soluionts

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$\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$. Thus, $\mathbf{B}^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$ so $\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$ so $\mathbf{x} = \begin{pmatrix} 0 \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$.

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$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ so $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Thus, $\mathbf{B}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ so $\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ so $\mathbf{x} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$.

$\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$. Thus, $\mathbf{B}^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$ so $\mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$ so $\mathbf{x} = \begin{pmatrix} 0 \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$.

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Principal Ideas

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Questions: How to choose first, next and last (optimal) extreme point?

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The standard linear programming problem

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Main idea

The *key idea* of the simplex method is to move from an extreme point to an improving adjacent extreme point by interchanging a column in **B** and a column in **N**.

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Checking for optimality

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Based on the derived expression for z, the *rate of change* of z with respect to the nonbasic variable x_j is:

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What is $(z_j - c_j)$ for a basic variable?

Determining the entering and departing variables

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Pick the non-basic variable for which $\frac{\partial z}{\partial x_j}$ is the largest.

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x_k is determined by a blocking constraint.

Forming a new basis

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Theorem

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Let $\mathbf{B} = (\mathbf{b}_1, \ \mathbf{b}_2, \ldots, \ \mathbf{b}_m)$ *be a basis for* E^m*, and let* $\mathbf{a} \in E^m$*,* $\mathbf{a} \neq \mathbf{0}$*. Then* $\mathbf{a}$ *can be written uniquely as a linear combination of* $\mathbf{b}_1, \ \mathbf{b}_2, \ldots, \ \mathbf{b}_m$*.*

Forming a new basis

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Example

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$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$

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$$\text{maximize } z = 2 \cdot x_1 + 3 \cdot x_2$$
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$$\begin{aligned} x_1 - 2 \cdot x_2 &\leq 4 \\ 2 \cdot x_1 + x_2 &\leq 18 \\ x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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Note

Solve the above problem graphically.

Standardization

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Standardizing the constraints

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This problem can be summarized as follows:

$$\begin{array}{rcl} \mathbf{A} & = & \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \\ \mathbf{B} & = & \begin{pmatrix} 4 \\ 18 \\ 10 \end{pmatrix} \\ \mathbf{c} & = & (2 \;\; 3 \;\; 0 \;\; 0 \;\; 0) \end{array}$$

Ploughing through

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Locate the initial basis

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An obvious choice is $\mathbf{I}$. $\mathbf{B} = (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$

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Is this basis feasible?

Moving from one basis to the next

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Basic variables in terms of non-basic variables

Expressing z and $\mathbf{x_B}$ in terms of $\mathbf{x_N}$, we get:

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Expressing z and $\mathbf{x_B}$ in terms of $\mathbf{x_N}$, we get:

$$\begin{aligned} z &= 2 \cdot x_1 + 3 \cdot x_2 \\ x_3 &= 4 - x_1 + x_2 \\ x_4 &= 18 - 2 \cdot x_1 - x_2 \\ x_5 &= 10 - x_2 \end{aligned}$$

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Is the current basic solution optimal?

Choosing the departing variables

Choosing the departing variables

Choosing the entering variable

$\partial z / \partial x_1 = 2.$

Choosing the departing variables

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x_5 is the *departing variable* and the corresponding constant is called the blocking constraint.

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The new canonically representation of z and $\mathbf{x}_B$ is are formed using $x_2 = 10 - x_5$ to eliminate x_2; i.e., to represent the basic variables x_2, x_3 and x_4 by the non-basic variables x_1 and x_5.

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$$
\begin{aligned}
z = \quad & 2 \cdot x_1 + 3 \cdot (10 - x_5) = 30 + 2 \cdot x_1 - 3 \cdot x_5 \\
x_3 = \quad & 4 - x_1 + 2 \cdot (10 - x_5) = 24 - x_1 - 2 \cdot x_5 \\
x_4 = \quad & 18 - 2 \cdot x_1 - (10 - x_5) = 8 - 2 \cdot x_1 + x_5 \\
x_2 = \quad & 10 - x_5
\end{aligned}
$$

New basis

Summary

The current solution and basis matrix can be summarized as follows:

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z &= 30 \\
\mathbf{x_B} &= \begin{pmatrix} x_{B,1} \\ x_{B_2} \\ x_{B_3} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ x_2 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \\ 10 \end{pmatrix} \\
\mathbf{x_N} &= \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\mathbf{B} &= (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_2) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
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$$

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Is the current solution optimal?

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\end{aligned}$$

Is the current solution optimal? Clearly not, since $\partial z / \partial x_1 = 2 \geq 0$. This also means that x_1 is the entering variable.

Final move

Final move

Departing variable

$$
\begin{aligned}
z = \quad & 30 + 2 \cdot x_1 - 3 \cdot x_5 \\
x_3 = \quad & 24 - x_1 - 2 \cdot x_5 \\
x_4 = \quad & 8 - 2 \cdot x_1 + x_5 \\
x_2 = \quad & 10 - x_5
\end{aligned}
$$

Clearly, $x_4 = 8 - 2 \cdot x_1 + x_5$ is the blocking constraint.
Thus x_1 can be raised up to 4. x_4 is now the departing variable.

Replacing x_1 with $4 - \frac{1}{2} \cdot x_4 + \frac{1}{2} \cdot x_5$, we get,

$$
\begin{aligned}
z = \quad & 30 + 2 \cdot (4 - \tfrac{1}{2} \cdot x_4 - \tfrac{1}{2} \cdot x_5) - 3 \cdot x_5 = 38 - x_4 - 2 \cdot x_5 \\
x_3 = \quad & 24 - (4 + \tfrac{1}{2} \cdot x_5 - \tfrac{1}{2} \cdot x_4) - 2 \cdot x_5 = 20 + \tfrac{1}{2} \cdot x_4 - \tfrac{5}{2} \cdot x_5 \\
x_1 = \quad & 4 - \tfrac{1}{2} x_4 + \tfrac{1}{2} x_5 \\
x_2 = \quad & 10 - x_5
\end{aligned}
$$

Final move

Departing variable

$$
\begin{aligned}
z &= 30 + 2 \cdot x_1 - 3 \cdot x_5 \\
x_3 &= 24 - x_1 - 2 \cdot x_5 \\
x_4 &= 8 - 2 \cdot x_1 + x_5 \\
x_2 &= 10 - x_5
\end{aligned}
$$

Clearly, $x_4 = 8 - 2 \cdot x_1 + x_5$ is the blocking constraint.
Thus x_1 can be raised up to 4. x_4 is now the departing variable.

Replacing x_1 with $4 - \frac{1}{2} \cdot x_4 + \frac{1}{2} \cdot x_5$, we get,

$$
\begin{aligned}
z &= 30 + 2 \cdot (4 - \tfrac{1}{2} \cdot x_4 - \tfrac{1}{2} \cdot x_5) - 3 \cdot x_5 = 38 - x_4 - 2 \cdot x_5 \\
x_3 &= 24 - (4 + \tfrac{1}{2} \cdot x_5 - \tfrac{1}{2} \cdot x_4) - 2 \cdot x_5 = 20 + \tfrac{1}{2} \cdot x_4 - \tfrac{5}{2} \cdot x_5 \\
x_1 &= 4 - \tfrac{1}{2} x_4 + \tfrac{1}{2} x_5 \\
x_2 &= 10 - x_5
\end{aligned}
$$

Is the new solution optimal?

Important observations

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1. *There is finite progress being made at each pivot.*

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Important observations

Note

1. *There is finite progress being made at each pivot. If the optimum is finite, the algorithm will converge (barring cycling).*
2. *How do we check for unboundedness?*
3. *How do we get the initial bfs?*

The product mix problem

The product mix problem

Example

The product mix problem

Example

We have two gadgets to produce: α and β.

The product mix problem

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1. The return for a unit of α is \$20.

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2. Each unit of α requires 4 hours of assembly and 1 hour of testing.
3. The return for a unit of β is \$30.

The product mix problem

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We have two gadgets to produce: α and β.

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The product mix problem

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5. We must produce at least 25 units of α.

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4. Each unit of β requires 3 hours of assembly and 2 hours of testing.
5. We must produce at least 25 units of α.
6. We have a total of 240 hours available for assembly and 140 hours for testing.

The product mix problem

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5. We must produce at least 25 units of α.
6. We have a total of 240 hours available for assembly and 140 hours for testing.

How many units of α and β should be produced to maximize our return?

Modeling the product mix problem

Modeling the product mix problem

Decision Variables

Modeling the product mix problem

Decision Variables

Let x_1 denote the number of units of α and x_2 denote the number of units of β to be manufactured.

Modeling the product mix problem

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Let x_1 denote the number of units of α and x_2 denote the number of units of β to be manufactured.

Objective function

Modeling the product mix problem

Decision Variables

Let x_1 denote the number of units of α and x_2 denote the number of units of β to be manufactured.

Objective function

$\max 20 \cdot x_1 + 30 \cdot x_2.$

Modeling the product mix problem

Decision Variables

Let x_1 denote the number of units of α and x_2 denote the number of units of β to be manufactured.

Objective function

$\max 20 \cdot x_1 + 30 \cdot x_2.$

Constraints

Modeling the product mix problem

Decision Variables

Let x_1 denote the number of units of α and x_2 denote the number of units of β to be manufactured.

Objective function

$\max 20 \cdot x_1 + 30 \cdot x_2$.

Constraints

Modeling the product mix problem

Decision Variables

Let x_1 denote the number of units of α and x_2 denote the number of units of β to be manufactured.

Objective function

$\max 20 \cdot x_1 + 30 \cdot x_2$.

Constraints

$$4 \cdot x_1 + 3 \cdot x_2 \leq 240$$

Modeling the product mix problem

Decision Variables

Let x_1 denote the number of units of α and x_2 denote the number of units of β to be manufactured.

Objective function

$\max 20 \cdot x_1 + 30 \cdot x_2$.

Constraints

$$\begin{aligned} 4 \cdot x_1 + 3 \cdot x_2 &\leq 240 \\ x_1 + 2 \cdot x_2 &\leq 140 \end{aligned}$$

Modeling the product mix problem

Decision Variables

Let x_1 denote the number of units of α and x_2 denote the number of units of β to be manufactured.

Objective function

$\max 20 \cdot x_1 + 30 \cdot x_2.$

Constraints

$$\begin{aligned} 4 \cdot x_1 + 3 \cdot x_2 &\leq 240 \\ x_1 + 2 \cdot x_2 &\leq 140 \\ x_1 &\geq 25 \end{aligned}$$

Modeling the product mix problem

Decision Variables

Let x_1 denote the number of units of α and x_2 denote the number of units of β to be manufactured.

Objective function

$\max 20 \cdot x_1 + 30 \cdot x_2$.

Constraints

$$\begin{aligned} 4 \cdot x_1 + 3 \cdot x_2 &\leq 240 \\ x_1 + 2 \cdot x_2 &\leq 140 \\ x_1 &\geq 25 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Portfolio optimization

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We want to invest $\$50,000$ among three strategies: savings certificates, municipal bonds, and stocks.

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4. The investment in stocks should not exceed the combined total investment in the other two strategies.
5. The savings certificate investment should be between $\$5,000$ and $\$15,000$.

How should we invest the money in order to maximize our return?

Modeling the portfolio optimization problem

Modeling the portfolio optimization problem

Decision Variables

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Let x_1, x_2 and x_3 denote the amounts to be invested in savings certificates, municipal bonds and stocks respectively.

Modeling the portfolio optimization problem

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Objective Function

Modeling the portfolio optimization problem

Decision Variables

Let x_1, x_2 and x_3 denote the amounts to be invested in savings certificates, municipal bonds and stocks respectively.

Objective Function

$\max 0.07 \cdot x_1 + 0.09 \cdot x_2 + 0.14 \cdot x_3$.

Modeling the portfolio optimization problem

Decision Variables

Let x_1, x_2 and x_3 denote the amounts to be invested in savings certificates, municipal bonds and stocks respectively.

Objective Function

$\max 0.07 \cdot x_1 + 0.09 \cdot x_2 + 0.14 \cdot x_3$.

Constraints

Modeling the portfolio optimization problem

Decision Variables

Let x_1, x_2 and x_3 denote the amounts to be invested in savings certificates, municipal bonds and stocks respectively.

Objective Function

$\max 0.07 \cdot x_1 + 0.09 \cdot x_2 + 0.14 \cdot x_3.$

Constraints

Modeling the portfolio optimization problem

Decision Variables

Let x_1, x_2 and x_3 denote the amounts to be invested in savings certificates, municipal bonds and stocks respectively.

Objective Function

$\max 0.07 \cdot x_1 + 0.09 \cdot x_2 + 0.14 \cdot x_3$.

Constraints

$$x_2 \geq 10,000$$

Modeling the portfolio optimization problem

Decision Variables

Let x_1, x_2 and x_3 denote the amounts to be invested in savings certificates, municipal bonds and stocks respectively.

Objective Function

$\max 0.07 \cdot x_1 + 0.09 \cdot x_2 + 0.14 \cdot x_3$.

Constraints

$$\begin{aligned} x_2 &\geq 10,000 \\ x_3 &\leq x_1 + x_2 \end{aligned}$$

Modeling the portfolio optimization problem

Decision Variables

Let x_1, x_2 and x_3 denote the amounts to be invested in savings certificates, municipal bonds and stocks respectively.

Objective Function

$\max 0.07 \cdot x_1 + 0.09 \cdot x_2 + 0.14 \cdot x_3$.

Constraints

$$\begin{aligned} x_2 &\geq 10,000 \\ x_3 &\leq x_1 + x_2 \\ x_1 &\geq 5000 \end{aligned}$$

Modeling the portfolio optimization problem

Decision Variables

Let x_1, x_2 and x_3 denote the amounts to be invested in savings certificates, municipal bonds and stocks respectively.

Objective Function

$\max 0.07 \cdot x_1 + 0.09 \cdot x_2 + 0.14 \cdot x_3$.

Constraints

$$\begin{aligned} x_2 &\geq 10,000 \\ x_3 &\leq x_1 + x_2 \\ x_1 &\geq 5000 \\ x_1 &\leq 15,000 \end{aligned}$$

Modeling the portfolio optimization problem

Decision Variables

Let x_1, x_2 and x_3 denote the amounts to be invested in savings certificates, municipal bonds and stocks respectively.

Objective Function

$\max 0.07 \cdot x_1 + 0.09 \cdot x_2 + 0.14 \cdot x_3.$

Constraints

$$
\begin{aligned}
x_2 &\geq 10,000 \\
x_3 &\leq x_1 + x_2 \\
x_1 &\geq 5000 \\
x_1 &\leq 15,000 \\
x_1 + x_2 + x_3 &\leq 50,000 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

Farmland Use

Farmland Use

Example

Farmland Use

Example

We own 500 acres of land, in which we grow corn, wheat, soybeans and oats.

Farmland Use

Example

We own 500 acres of land, in which we grow corn, wheat, soybeans and oats.

1. An acre yields 110 bushels of corn,

Farmland Use

Example

We own 500 acres of land, in which we grow corn, wheat, soybeans and oats.

1. An acre yields 110 bushels of corn, 35 bushels of wheat,

Farmland Use

Example

We own 500 acres of land, in which we grow corn, wheat, soybeans and oats.

1. An acre yields 110 bushels of corn, 35 bushels of wheat, 32 bushels of soybeans,

Farmland Use

Example

We own 500 acres of land, in which we grow corn, wheat, soybeans and oats.

1. An acre yields 110 bushels of corn, 35 bushels of wheat, 32 bushels of soybeans, and 55 bushels of oats.

Farmland Use

Example

We own 500 acres of land, in which we grow corn, wheat, soybeans and oats.

1. An acre yields 110 bushels of corn, 35 bushels of wheat, 32 bushels of soybeans, and 55 bushels of oats.
2. To receive federal subsidies, we may not plant more than 120 acres of soybeans.

Farmland Use

Example

We own 500 acres of land, in which we grow corn, wheat, soybeans and oats.

1. An acre yields 110 bushels of corn, 35 bushels of wheat, 32 bushels of soybeans, and 55 bushels of oats.
2. To receive federal subsidies, we may not plant more than 120 acres of soybeans.
3. We require at least $10,000$ bushels of corn product due to a contract with a local dairy farm.

Farmland Use

Example

We own 500 acres of land, in which we grow corn, wheat, soybeans and oats.

1. An acre yields 110 bushels of corn, 35 bushels of wheat, 32 bushels of soybeans, and 55 bushels of oats.
2. To receive federal subsidies, we may not plant more than 120 acres of soybeans.
3. We require at least $10,000$ bushels of corn product due to a contract with a local dairy farm.
4. The total wheat acreage should not be less than that used for soybeans and oats.

Farmland Use

Example

We own 500 acres of land, in which we grow corn, wheat, soybeans and oats.

1. An acre yields 110 bushels of corn, 35 bushels of wheat, 32 bushels of soybeans, and 55 bushels of oats.
2. To receive federal subsidies, we may not plant more than 120 acres of soybeans.
3. We require at least $10,000$ bushels of corn product due to a contract with a local dairy farm.
4. The total wheat acreage should not be less than that used for soybeans and oats.
5. The selling price per bushel of corn is \$0.36; of wheat, \$0.90; of soybeans, \$0.82; of oats, \$0.98.

Farmland Use

Example

We own 500 acres of land, in which we grow corn, wheat, soybeans and oats.

1. An acre yields 110 bushels of corn, 35 bushels of wheat, 32 bushels of soybeans, and 55 bushels of oats.
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5. The selling price per bushel of corn is \$0.36; of wheat, \$0.90; of soybeans, \$0.82; of oats, \$0.98.

How many acres of each product should be grown to maximize our profit?

Modeling the Farmland Use problem

Modeling the Farmland Use problem

Decision Variables

Modeling the Farmland Use problem

Decision Variables

Let x_1, x_2, x_3 and x_4 denote the acreage of corn, what, soybeans and oats respectively.

Modeling the Farmland Use problem

Decision Variables

Let x_1, x_2, x_3 and x_4 denote the acreage of corn, what, soybeans and oats respectively.

Objective Function

Modeling the Farmland Use problem

Decision Variables

Let x_1, x_2, x_3 and x_4 denote the acreage of corn, what, soybeans and oats respectively.

Objective Function

$$\max(0.36) \cdot 110 \cdot x_1 + (0.9) \cdot 35 \cdot x_2 + (0.82) \cdot 32 \cdot x_3 + (0.98) \cdot 55 \cdot x_4.$$

Modeling the Farmland Use problem

Decision Variables

Let x_1, x_2, x_3 and x_4 denote the acreage of corn, what, soybeans and oats respectively.

Objective Function

$\max(0.36) \cdot 110 \cdot x_1 + (0.9) \cdot 35 \cdot x_2 + (0.82) \cdot 32 \cdot x_3 + (0.98) \cdot 55 \cdot x_4.$

Constraints

Modeling the Farmland Use problem

Decision Variables

Let x_1, x_2, x_3 and x_4 denote the acreage of corn, what, soybeans and oats respectively.

Objective Function

$\max(0.36) \cdot 110 \cdot x_1 + (0.9) \cdot 35 \cdot x_2 + (0.82) \cdot 32 \cdot x_3 + (0.98) \cdot 55 \cdot x_4.$

Constraints

Modeling the Farmland Use problem

Decision Variables

Let x_1, x_2, x_3 and x_4 denote the acreage of corn, what, soybeans and oats respectively.

Objective Function

$\max(0.36) \cdot 110 \cdot x_1 + (0.9) \cdot 35 \cdot x_2 + (0.82) \cdot 32 \cdot x_3 + (0.98) \cdot 55 \cdot x_4.$

Constraints

$$x_1 + x_2 + x_3 + x_4 \leq 500$$

Modeling the Farmland Use problem

Decision Variables

Let x_1, x_2, x_3 and x_4 denote the acreage of corn, what, soybeans and oats respectively.

Objective Function

$\max(0.36) \cdot 110 \cdot x_1 + (0.9) \cdot 35 \cdot x_2 + (0.82) \cdot 32 \cdot x_3 + (0.98) \cdot 55 \cdot x_4.$

Constraints

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &\leq 500 \\ x_3 &\leq 120 \end{aligned}$$

Modeling the Farmland Use problem

Decision Variables

Let x_1, x_2, x_3 and x_4 denote the acreage of corn, what, soybeans and oats respectively.

Objective Function

$\max(0.36) \cdot 110 \cdot x_1 + (0.9) \cdot 35 \cdot x_2 + (0.82) \cdot 32 \cdot x_3 + (0.98) \cdot 55 \cdot x_4.$

Constraints

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &\leq 500 \\
x_3 &\leq 120 \\
110 \cdot x_1 &\geq 10,000
\end{aligned}
$$

Modeling the Farmland Use problem

Decision Variables

Let x_1, x_2, x_3 and x_4 denote the acreage of corn, what, soybeans and oats respectively.

Objective Function

$\max (0.36) \cdot 110 \cdot x_1 + (0.9) \cdot 35 \cdot x_2 + (0.82) \cdot 32 \cdot x_3 + (0.98) \cdot 55 \cdot x_4.$

Constraints

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &\leq 500 \\ x_3 &\leq 120 \\ 110 \cdot x_1 &\geq 10,000 \\ x_2 &\geq x_3 + x_4 \end{aligned}$$

Modeling the Farmland Use problem

Decision Variables

Let x_1, x_2, x_3 and x_4 denote the acreage of corn, what, soybeans and oats respectively.

Objective Function

$\max(0.36) \cdot 110 \cdot x_1 + (0.9) \cdot 35 \cdot x_2 + (0.82) \cdot 32 \cdot x_3 + (0.98) \cdot 55 \cdot x_4$.

Constraints

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &\leq 500 \\
x_3 &\leq 120 \\
110 \cdot x_1 &\geq 10,000 \\
x_2 &\geq x_3 + x_4 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

Transportation

Transportation

Example

Transportation

Example

We have three warehouses and four clients.

Transportation

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We have three warehouses and four clients.

1. Warehouses 1, 2, and 3 have $6,000$, $9,000$, and $4,000$ units available respectively.

Transportation

Example

We have three warehouses and four clients.

1. Warehouses 1, 2, and 3 have $6,000$, $9,000$, and $4,000$ units available respectively.
2. Clients 1, 2, 3, and 4 want $3,900$, $5,200$, $2,700$, and $6,400$ units respectively.

Transportation

Example

We have three warehouses and four clients.

1. Warehouses 1, 2, and 3 have $6,000$, $9,000$, and $4,000$ units available respectively.
2. Clients 1, 2, 3, and 4 want $3,900$, $5,200$, $2,700$, and $6,400$ units respectively.
3. The cost to ship a unit from a given warehouse to a given client varies according to the following table:

Transportation

Example

We have three warehouses and four clients.

1. Warehouses 1, 2, and 3 have 6,000, 9,000, and 4,000 units available respectively.
2. Clients 1, 2, 3, and 4 want 3,900, 5,200, 2,700, and 6,400 units respectively.
3. The cost to ship a unit from a given warehouse to a given client varies according to the following table:

	Client			
Warehouse	1	2	3	4
1	7	3	8	4
2	8	5	6	3
3	4	6	9	6

Transportation

Example

We have three warehouses and four clients.

1. Warehouses 1, 2, and 3 have $6,000$, $9,000$, and $4,000$ units available respectively.
2. Clients 1, 2, 3, and 4 want $3,900$, $5,200$, $2,700$, and $6,400$ units respectively.
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	Client			
Warehouse	1	2	3	4
1	7	3	8	4
2	8	5	6	3
3	4	6	9	6

4. Items should be shipped from warehouses to clients, so all client demands are met.

Transportation

Example

We have three warehouses and four clients.

1. Warehouses 1, 2, and 3 have 6,000, 9,000, and 4,000 units available respectively.
2. Clients 1, 2, 3, and 4 want 3,900, 5,200, 2,700, and 6,400 units respectively.
3. The cost to ship a unit from a given warehouse to a given client varies according to the following table:

	Client			
Warehouse	1	2	3	4
1	7	3	8	4
2	8	5	6	3
3	4	6	9	6

4. Items should be shipped from warehouses to clients, so all client demands are met.

How can we perform the shipping while minimizing our shipping cost?

Modeling the Transportation problem

Modeling the Transportation problem

Decision Variables

Modeling the Transportation problem

Decision Variables

Let $x_{i,j}$ denote the number of units to be shipped from warehouse i to client j, where $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$.

Modeling the Transportation problem

Decision Variables

Let $x_{i,j}$ denote the number of units to be shipped from warehouse i to client j, where $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$.

Objective Function

Modeling the Transportation problem

Decision Variables

Let $x_{i,j}$ denote the number of units to be shipped from warehouse i to client j, where $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$.

Objective Function

$$\begin{aligned} \min \quad & 7 \cdot x_{1,1} + 3 \cdot x_{1,2} + 8 \cdot x_{1,3} + 4 \cdot x_{1,4} \\ & 9 \cdot x_{2,1} + 5 \cdot x_{2,2} + 6 \cdot x_{2,3} + 3 \cdot x_{2,4} \\ & 4 \cdot x_{3,1} + 6 \cdot x_{3,2} + 9 \cdot x_{3,3} + 6 \cdot x_{3,4} \end{aligned}$$

Modeling (contd.)

Modeling (contd.)

Constraints

Modeling (contd.)

Constraints

The supply constraints:

Modeling (contd.)

Constraints

The supply constraints:

$$\sum_{j=1}^{4} x_{1,j} \leq 6000$$

Modeling (contd.)

Constraints

The supply constraints:

$$\sum_{j=1}^{4} x_{1,j} \leq 6000$$

$$\sum_{j=1}^{4} x_{2,j} \leq 9000$$

Modeling (contd.)

Constraints

The supply constraints:

$$\sum_{j=1}^{4} x_{1,j} \leq 6000$$

$$\sum_{j=1}^{4} x_{2,j} \leq 9000$$

$$\sum_{j=1}^{4} x_{3,j} \leq 4000$$

Modeling (contd.)

Constraints

The supply constraints:

$$\sum_{j=1}^{4} x_{1,j} \leq 6000$$

$$\sum_{j=1}^{4} x_{2,j} \leq 9000$$

$$\sum_{j=1}^{4} x_{3,j} \leq 4000$$

The demand constraints:

$$\sum_{i=1}^{3} x_{i,1} = 3900$$

$$\sum_{i=1}^{3} x_{i,2} = 5200$$

$$\sum_{i=1}^{3} x_{i,3} = 2700$$

$$\sum_{i=1}^{3} x_{i,4} = 6400$$

Modeling (contd.)

Constraints

The supply constraints:

$$\sum_{j=1}^{4} x_{1,j} \leq 6000$$
$$\sum_{j=1}^{4} x_{2,j} \leq 9000$$
$$\sum_{j=1}^{4} x_{3,j} \leq 4000$$

The demand constraints:

$$\sum_{i=1}^{3} x_{i,1} = 3900$$
$$\sum_{i=1}^{3} x_{i,2} = 5200$$
$$\sum_{i=1}^{3} x_{i,3} = 2700$$
$$\sum_{i=1}^{3} x_{i,4} = 6400$$

Non-negativity constraints:
$x_{ij} \geq 0$, $i = 1, 2, 3$, $j = 1, 2, 3, 4$.

Uncovered topics

Uncovered topics

Self-study

Uncovered topics

Self-study

1. The Simplex tableau method.

Uncovered topics

Self-study

1. The Simplex tableau method.
2. Degeneracy and cycling.

Uncovered topics

Self-study

1. The Simplex tableau method.
2. Degeneracy and cycling.
3. The revised simplex method.

Uncovered topics

Self-study

1. The Simplex tableau method.
2. Degeneracy and cycling.
3. The revised simplex method.
4. The bounded variables simplex method.

Uncovered topics

Self-study

1. The Simplex tableau method.
2. Degeneracy and cycling.
3. The revised simplex method.
4. The bounded variables simplex method.
5. Decomposition.

Uncovered topics

Self-study

1. The Simplex tableau method.
2. Degeneracy and cycling.
3. The revised simplex method.
4. The bounded variables simplex method.
5. Decomposition.
6. Sensitivity analysis.

Alternatives to the Simplex Method

Alternatives to the Simplex Method

Issues and Alternatives

Alternatives to the Simplex Method

Issues and Alternatives

1. Computational Complexity.

Alternatives to the Simplex Method

Issues and Alternatives

1. Computational Complexity.
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