Mathematical Preliminaries

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Outline

1. Linear Algebra
   - Vectors
   - Matrices
   - The Solution of Simultaneous Linear Equations

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2. Convexity and Cones
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   - Sample Space and Events
   - Defining Probabilities on Events
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   - Concentration Inequalities

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Basics

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The collection of all m-dimensional vectors is called **Euclidean m-space** and is denoted by E^m (also $\Re^m$).
Vectors can be represented geometrically, where a vector can be thought of as either a point or as an arrow directed from the origin to the point.

Example

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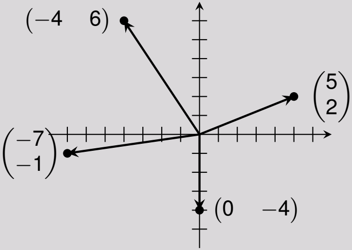
Euclidean 2-space, E^2.



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Vector addition satisfies both the commutative ($\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$) and associative ($\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + \mathbf{b} + \mathbf{c}$) laws.

Vector Addition Example

$$\mathbf{a} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 9 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 6 & 8 & 0 \end{pmatrix} \quad \mathbf{d} = \begin{pmatrix} 4 \\ 10 \\ 2 \\ 3 \end{pmatrix}$$

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$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} + \begin{pmatrix} 5 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 8 \end{pmatrix}$$

$\mathbf{a} + \mathbf{c}$ is undefined (not the same type)
$\mathbf{a} + \mathbf{d}$ is undefined (different number of elements)

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$$\alpha \cdot \mathbf{b} = \alpha \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} \alpha \cdot b_1 \\ \alpha \cdot b_2 \\ \vdots \\ \alpha \cdot b_m \end{pmatrix}$$

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We should also note that vector multiplication satisfies the distributive law $\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac}$.

Vectors

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$$\mathbf{a} = \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix}$$

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$$\mathbf{ca} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 7 \end{pmatrix} = 12 + 0 + 14 = 26$$

$$\mathbf{cb} = \begin{pmatrix} 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 10 \\ 1 \end{pmatrix} = -8 + 90 + 2 = 84$$

Norms

Norm of a Vector

The $\mathbf{L_p}$ norm of a vector $\mathbf{a} \in E^n$, denoted by $\|\mathbf{a}\|_p$, is a measure of the size of $\mathbf{a}$ and is given by $\|\mathbf{a}\|_p = \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}$.

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$$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \quad \|\mathbf{a}\|_2 = [3^2 + 2^2 + (-1)^2]^{1/2} = (14)^{1/2}$$

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Note

The dot product of two vectors can also be defined by using the Euclidean norm, which is given by $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|_2 \cdot \|\mathbf{b}\|_2 \cos\theta$, where θ is the angle between the two vectors.

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Special Vector Types

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Linear Dependence and Independence

A set of vectors, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is **linearly dependent** if there exist some scalars, α_i, that are not all zero such that

$$\alpha_1 \cdot \mathbf{a}_1 + \alpha_2 \cdot \mathbf{a}_2 + \cdots + \alpha_m \cdot \mathbf{a}_m = \mathbf{0} \quad (1)$$

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If the only set of scalars, α_i, for which the above equation holds is $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, the vectors are **linearly independent**.

Example

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Linearly Dependent:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \mathbf{a}_3 = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

$$2\mathbf{a}_1 + 3\mathbf{a}_2 - 1\mathbf{a}_3 = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 3 \end{pmatrix} - 1\begin{pmatrix} 8 \\ 11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Linearly Independent:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Consider the equation

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\alpha_1 + 2\alpha_2 = 0 \tag{2}$$

$$\alpha_1 = 0 \tag{3}$$

We can see that the only solution is $\alpha_1 = \alpha_2 = 0$. This means $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent.

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Spanning Sets and Bases

The vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p \in E^n$ are said to form a **spanning set** if every vector in E^n can be written as a linear combination of the $\mathbf{b}_i$.

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In other words, if $\mathbf{v} \in E^n$, then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_p$ such that $\mathbf{v} = \alpha_1 \cdot \mathbf{b}_1 + \alpha_2 \cdot \mathbf{b}_2 + \cdots + \alpha_p \cdot \mathbf{b}_p$.

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We say that the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \in E^n$ form a **basis** for E^n, if they are linearly independent and form a spanning set for E^n.

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Note that a basis is a minimal spanning set. This is because adding a new vector would make the set linearly dependent and removing one of the vectors would mean the remaining ones no longer span E^n.

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Example

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

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Example

$$\mathbf{A} = \begin{pmatrix} 7 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 5 & 9 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 2 & 1 \\ 7 & 3 \\ 9 & 2 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 7 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -3 & 4 \\ 1 & 5 & 9 \end{pmatrix} = \begin{pmatrix} 9 & -2 & 2 \\ 4 & 8 & 9 \end{pmatrix}$$

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Example

$$\beta = 3 \quad \mathbf{A} = \begin{pmatrix} 8 & 3 \\ -1 & 2 \\ 7 & 1 \end{pmatrix} \quad \beta \cdot \mathbf{A} = 3 \begin{pmatrix} 8 & 3 \\ -1 & 2 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} 24 & 9 \\ -3 & 6 \\ 21 & 3 \end{pmatrix}$$

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Each element in **C** is given by $c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$, where n is the number of columns of **A** or rows of **B**, $i = 1, \ldots, m$ where m is the number of rows of **A**, and $j = 1, \ldots, q$ where q is the number of columns of **B**.

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Matrix multiplication satisfies the associative and distributive laws, but it does **not** satisfy the commutative law in general.

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$$\mathbf{A} = \begin{pmatrix} 7 & 1 \\ 4 & -3 \\ 2 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & 7 \\ 0 & -1 & 4 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 7 & 1 \\ 4 & -3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 7 \\ 0 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 14 & 6 & 53 \\ 8 & 7 & 16 \\ 4 & 2 & 14 \end{pmatrix}$$

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$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & 4 & 9 \end{pmatrix}$$

Positive Semidefinite - A symmetric matrix $\mathbf{A}$ is said to be positive semidefinite, if $\mathbf{x}^{\mathbf{T}} \cdot \mathbf{A} \cdot \mathbf{x} \geq 0$ for all $\mathbf{x}$ and $\mathbf{x}^{\mathbf{T}} \cdot A \cdot \mathbf{x} = 0$, only if $\mathbf{x} = \mathbf{0}$.

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$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 5 & 6 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & 2 \\ 1 & 9 \end{pmatrix} \quad (\mathbf{A}|\mathbf{B}) = \left(\begin{array}{cc|cc} 1 & 4 & 3 & 2 \\ 5 & 6 & 1 & 9 \end{array} \right)$$

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The **cofactor** of an element is its minor with the sign $(-1)^{i+j}$ attached to it.

Example

$$|\mathbf{A}| = \begin{vmatrix} 7 & -1 & 0 \\ 3 & 2 & 1 \\ 8 & 1 & -4 \end{vmatrix}$$

The cofactor for $a_{2,1} = 3$ is

$$(-1)^{2+1}|\mathbf{A}_{2,1}| = (-1)\begin{vmatrix} -1 & 0 \\ 1 & -4 \end{vmatrix} = -4$$

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$$|\mathbf{A}| = \sum_{j=1}^{n} a_{i,j}(-1)^{i+j}|\mathbf{A}_{i,j}|$$

and for any column j, this would be

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Expanding $|\mathbf{A}|$ by column 3, we get

$$\begin{aligned} |\mathbf{A}| &= 3(-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} + 2(-1)^{2+3}\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + 5(-1)^{3+3}\begin{vmatrix} 1 & 4 \\ 2 & 0 \end{vmatrix} \\ &= 3(6) - 2(-1) + 5(-8) = -20 \end{aligned}$$

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Value of Determinants (Contd.)

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5. To the elements of any row, you may add a constant times the corresponding element of any other row without changing the value of the determinant.

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Let $\gamma_{i,j} = (-1)^{i+j}|\mathbf{A}_{i,j}|$ be the cofactor for $a_{i,j}$, then

$$\mathbf{A}^\alpha = \begin{pmatrix} \gamma_{1,1} & \gamma_{2,1} & \cdots & \gamma_{n,1} \\ \gamma_{1,2} & \gamma_{2,2} & \cdots & \gamma_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1,n} & \gamma_{2,n} & \cdots & \gamma_{n,n} \end{pmatrix}$$

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Given a nonsingular matrix **A**, we find the inverse by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{A}^{\alpha}$$

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2. Multiply a row *i* by a nonzero scalar α.
3. Replace a row *i* by a row *i* plus a multiple of some row *j*.

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This shows that $r(\mathbf{A}) = k$.

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This means that the rank of **A** is 2.

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   - Vectors
   - Matrices
   - The Solution of Simultaneous Linear Equations
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$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = & b_2 \\ & & & & & & & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = & b_m \end{array}$$

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$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

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1. If $r(\mathbf{A}|\mathbf{b}) = r(\mathbf{A}) + 1$, then no solution exists.
2. If $r(\mathbf{A}|\mathbf{b}) = r(\mathbf{A})$, then there does exist a solution. This is because we can write $\mathbf{b}$ as a linear combination of the columns of $\mathbf{A}$. Furthermore, if $r(\mathbf{A}) = n$, where n is the number of variables, then there exists a unique solution for the system of equations.

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Cramer's rule states that the unique solution is given by $x_j = \frac{|\mathbf{A}_j|}{|\mathbf{A}|}$, for all $j = 1, \ldots, n$.

Cramer's rule

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Using Cramer's Rule

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & 2x_3 & = & 6 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 9 \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$$

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

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Given two points $\mathbf{x}$ and $\mathbf{y}$ in E^m, and $\alpha \in [0, 1]$, the parametric point $\alpha \cdot \mathbf{x} + (1 - \alpha) \cdot \mathbf{y}$ is said to be a convex combination of $\mathbf{x}$ and $\mathbf{y}$.

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A set of the form $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ *is said to be a polyhedral set.*

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If $<$ holds as opposed to $\leq$, the function is said to be strictly convex.

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$$f(\lambda \cdot \mathbf{x} + (1 - \lambda) \cdot \mathbf{y}) \leq \lambda \cdot f(\mathbf{x}) + (1 - \lambda) \cdot f(\mathbf{y}).$$

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$$[\bigtriangledown^2 f(\mathbf{x})]_{i,j} = \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}, \ \forall i, j$$

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If S is a convex set and f is a convex function of **x** *on S, the all local optima are also global optima.*

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Are cones convex?

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Are cones convex? We will be dealing with pointed, convex cones only.

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4. The cone of symmetric positive semidefinite matrices - $\{\mathbf{X} \in \Re^{n \times n} : \mathbf{X} = \mathbf{X}^{\mathbf{T}}, \text{ and } \mathbf{X} \text{ is positive semidefinite}\}$.

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Exercise

Show that the cone $\Re^n_+$ is its own dual cone.

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Any subset of the sample space S is called an event.

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Let G be another event on S. What is $P(E \cup F \cup G)$?

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Consider the experiment of tossing two fair dice. Let F denote the event that the first die turns up 4. Let E_1 denote the event that the sum of the faces of the two dice is 6. Let E_2 denote the event that the sum of the faces of the two dice is 7.

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Thus, the probability of an event E is the weighted average of the conditional probability of E, given that event F has occurred and the conditional probability of E, given that event F has not occurred, each conditional probability being given as much weight as the probability of the event that it is conditioned on, has of occurring.

Outline

1. Linear Algebra
   - Vectors
   - Matrices
   - The Solution of Simultaneous Linear Equations
2. Convexity and Cones
   - Convexity
   - Cones
3. Probability and Expectation
   - Sample Space and Events
   - Defining Probabilities on Events
   - Conditional Probability
   - Random Variables
   - Concentration Inequalities
4. Basic optimization theory
   - Fundamentals
5. Models of Optimization
   - Tools of Optimization
6. Financial Mathematics
   - Quantitative models
   - Problem Types

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$$p(i) = P\{X = i\} = C(n, i) \cdot p^i \cdot (1-p)^{n-i}, \; i = 0, 1, 2, \ldots n$$

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For a random variable X, the distribution function $F(\cdot)$ is defined for any real number b, $-\infty < b < \infty$, by

$$F(b) = P(X \leq b).$$

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Parameters of the important Random Variables

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Parameter table

Variable type	Expectation	Variance
Bernoulli	p	$p \cdot (1-p)$
Binomial	$n \cdot p$	$n \cdot p \cdot (1-p)$
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Exercise

Find the parameters of the Poisson, Normal, Uniform and exponential random variables.

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$$Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2 \cdot Cov(X_1, X_2).$$

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Tail bounds

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Chebyshev	$E[X], Var(X)$	$P(\lvert X - E[X]\rvert \geq a \cdot E[X]) \leq \frac{Var(X)}{(a \cdot E[X])^2}, a > 0.$
Chernoff	X is binomial, $E[X]$	$P((X - E[X]) \geq \delta) \leq e^{-\frac{-2 \cdot \delta^2}{n}}, \delta > 0.$

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Exercise

Find the tail bounds for the coin tossing problem using all three techniques.

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Given a function $f : \Re^n \rightarrow \Re$ and a set $S \subseteq \Re^n$, the problem of finding an $x^* \in \Re^n$ that solves

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Models of Optimization

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Financial Mathematics

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Principal issues

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1. Modern finance has become extremely technical.

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Portfolio Selection and asset allocation

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Asset/liability Management

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