Non-linear Programming and Solver

Vahan Mkrtchyan [1]

[1] Lane Department of Computer Science and Electrical Engineering
West Virginia University

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Outline

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5. Non-smooth Optimization: Subgradient methods

Portfolio Optimization

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Problem formulation

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Now, assume that we are aiming for a return $R_\rho\%$, and that we want to pay the least charges to achieve this.

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The last inequalities are included because investments must obviously be positive.

If we tried to solve the problem without them, an optimization algorithm would attempt to reduce costs by making one or more of the y_i large and negative.

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As an example, consider the following data for three assets over six months.

Monthly rates of return on three assets

Assets/Months	January	February	March	April	May	June
Assets 1	1.2	1.3	1.4	1.5	1.1	1.2
Assets 2	1.3	1.0	0.8	0.9	1.4	1.3
Assets 3	0.9	1.1	1.0	1.1	1.1	1.3

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where r_{ij} denotes the return on asset i in period j.

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The expected portfolio return is given by

$$R = \sum_{i=1}^{n} \bar{r}_i \cdot y_i.$$

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while the covariance of assets i and k is

$$\sigma_{ik} = \frac{\sum_{j=1}^{m} (r_{ij} - \bar{r}_i) \cdot (r_{kj} - \bar{r}_k)}{m}.$$

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Portfolio variance

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The variance of the portfolio defined by the investment fractions y_1, ...,y_n is

$$V = \sum_{i=1}^{n} \sigma_i^2 \cdot y_i^2 + 2 \cdot \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sigma_{ij} \cdot y_i \cdot y_j,$$

which can be used as a measure of portfolio risk.

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The constraint on partitions can be written as

$$S = \mathbf{e} \cdot \mathbf{y},$$

where $\mathbf{e} = (1, 1, ..., 1)$.

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In a rather general way, we can say that an optimal portfolio is one which gives "biggest return at lowest risk".

One way of trying to determine such a portfolio is to consider a composite function such as

$$F = -R + \rho \cdot V = -\bar{\mathbf{r}} \cdot \mathbf{y} + \rho \cdot \mathbf{y}^T \cdot \mathbf{Q} \cdot \mathbf{y}.$$

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The positive constant ρ controls the balance between return and risk.

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Given a function $f : R^n \to R$ and a set $S \subseteq R^n$, the problem of finding an $\mathbf{x}^* \in R^n$ that solves

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Constrained or Unconstrained Optimization

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$$g_i(\mathbf{x}) \geq 0, i \in \mathcal{I}$$

Non-linear programs

Such optimization problems usually are called *non-linear programming problems* or *non-linear programs.*

Constrained or Unconstrained Optimization

A non-linear program in which the set $\mathcal{E} \cup \mathcal{I}$ is empty, is called an unconstrained program.

Non-linear programming

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A non-linear program in which the set $\mathcal{E} \cup \mathcal{I}$ is empty, is called an unconstrained program.

Otherwise, it is constrained.

Non-linear programs arise

Probabilistic elements

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Nonlinearities may arise when some of the coefficients in the model are random variables.

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Non-linear programs arise

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Nonlinearities may arise when some of the coefficients in the model are random variables.

For example, consider a linear program, where the right-hand sides are random.

We consider the case when we have two constraints.

Probabilistic elements in linear programming

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Problem Formulation

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Problem Formulation

$$\max c_1 \cdot x_1 + \ldots + c_n \cdot x_n$$

Probabilistic elements in linear programming

Problem Formulation

$$\max c_1 \cdot x_1 + \ldots + c_n \cdot x_n$$

$$a_{11} \cdot x_1 + \ldots + a_{1n} \cdot x_n \leq b_1$$

Probabilistic elements in linear programming

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$$\max c_1 \cdot x_1 + ... + c_n \cdot x_n$$
$$a_{11} \cdot x_1 + ... + a_{1n} \cdot x_n \leq b_1$$
$$a_{21} \cdot x_1 + ... + a_{2n} \cdot x_n \leq b_2$$

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where the coefficients b_1 and b_2 are independently distributed and $G_i(y)$ represents the probability that the random variable b_i is at least as large as y.

Probabilistic elements in linear programming

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Problem Formulation

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Problem Formulation

Assume that we would like to choose the variables $x_1, \ldots, x_n$, so that the joint probability of both of the constraints being satisfied is at least β.

Probabilistic elements in linear programming

Problem Formulation

Assume that we would like to choose the variables $x_1, ..., x_n$, so that the joint probability of both of the constraints being satisfied is at least β.

The last constraint mathematically can be written as follows:

$$Pr[a_{11} \cdot x_1 + ... + a_{1n} \cdot x_n \leq b_1] \times Pr[a_{21} \cdot x_1 + ... + a_{2n} \cdot x_n \leq b_2] \geq \beta.$$

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Then this condition can be written as the following set of constraints:

$$-y_1 + a_{11} \cdot x_1 + ... + a_{1n} \cdot x_n = 0$$

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$$G_1(y_1) \times G_2(y_2) \geq \beta.$$

Constrained Optimization

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Problem Formulation

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$$\min f(\mathbf{x})$$

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Here we assume that we have at least one constraint, i.e., the set $\mathcal{E} \cup \mathcal{I}$ is not empty.

Constrained Optimization

Problem Formulation

$$\begin{aligned} &\min f(\mathbf{x}) \\ g_i(\mathbf{x}) &= 0, i \in \mathcal{E} \\ g_i(\mathbf{x}) &\geq 0, i \in \mathcal{I} \end{aligned}$$

Here we assume that we have at least one constraint, i.e., the set $\mathcal{E} \cup \mathcal{I}$ is not empty.

Moreover, we assume that the functions f and g_i are continuously differentiable.

Constrained Optimization

Constrained Optimization

Lagrangian function of the problem

Constrained Optimization

Lagrangian function of the problem

The Lagrangian function (or Lagrangian) is defined as follows:

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \cdot g_i(\mathbf{x}).$$

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Why Lagrangian function?

Constrained Optimization

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Why Lagrangian function?

It turns out that for suitably chosen values of λ_i, minimizing the unconstrained Lagrangian function $L(\mathbf{x}, \lambda)$ is equivalent to solving the above constrained non-linear program.

Some definitions

Some definitions

Definition

Some definitions

Definition

A point $\mathbf{x}$ satisfying $g_i(\mathbf{x}) = 0, i \in \mathcal{E}$ and $g_i(\mathbf{x}) \geq 0, i \in \mathcal{I}$ is called a feasible solution to the non-linear program.

Some definitions

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Definition

Let $\mathbf{x}$ be a feasible solution to the non-linear program, and let $\mathcal{J} \subseteq \mathcal{I}$ be the set of indices for which $g_i(\mathbf{x}) \geq 0$ is satisfied with equality.

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Definition

Let **x** be a feasible solution to the non-linear program, and let $\mathcal{J} \subseteq \mathcal{I}$ be the set of indices for which $g_i(\mathbf{x}) \geq 0$ is satisfied with equality.

Then **x** is a regular point of the program, if the gradient vectors $\triangledown \mathbf{g_i}(\mathbf{x})$ for $i \in \mathcal{E} \cup \mathcal{J}$ are linearly independent.

Some examples

Some examples

Problem Formulation: Regular points

Some examples

Problem Formulation: Regular points

$$\max(x^2 + y^2)$$

Some examples

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$$\max(x^2 + y^2)$$
$$x \geq 0$$

Some examples

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$$y \geq 0$$
$$x + y \leq 1$$

Some examples

Problem Formulation: Regular points

$$\max(x^2 + y^2)$$
$$x \geq 0$$
$$y \geq 0$$
$$x + y \leq 1$$

In this example any feasible point is regular, since the gradients of the constraints are $(1, 0)$, $(0, 1)$ and $(-1, -1)$.

Some examples

Some examples

Problem Formulation: Non-regular points

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$$\max(x^2 + y^2)$$

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Some examples

Problem Formulation: Non-regular points

$$\max(x^2 + y^2)$$
$$x \geq 0$$
$$y \geq 0$$
$$x^3 + 3 \cdot y^2 \leq 0$$

Some examples

Problem Formulation: Non-regular points

$$\max(x^2 + y^2)$$
$$x \geq 0$$
$$y \geq 0$$
$$x^3 + 3 \cdot y^2 \leq 0$$

In this example the feasible solution $(0, 0)$ is not a regular point since the gradients of the constraints are $(1, 0)$, $(0, 1)$ and $(0, 0)$.

First order necessary conditions

First order necessary conditions

KKT conditions

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KKT conditions

The conditions that will be presented in the upcoming three theorems are called Karush-Kuhn-Tucker (KKT) conditions after their inventors.

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Let $\mathbf{x}^*$ *be a local minimizer of the non-linear problem,*

First order necessary conditions

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The conditions that will be presented in the upcoming three theorems are called Karush-Kuhn-Tucker (KKT) conditions after their inventors.

Theorem

Let $\mathbf{x}^*$ *be a local minimizer of the non-linear problem, and assume that* $\mathbf{x}^*$ *is a regular point for the constraints of the problem.*

First order necessary conditions

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Theorem

Let $\mathbf{x}^*$ be a local minimizer of the non-linear problem, and assume that $\mathbf{x}^*$ is a regular point for the constraints of the problem.

Then there exists $\lambda_i, i \in \mathcal{E} \cup \mathcal{I}$ such that

$$\triangledown f(x^*) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \cdot \triangledown g_i(x^*) = 0,$$

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$$\lambda_i \geq 0, i \in \mathcal{I},$$

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Then there exists $\lambda_i, i \in \mathcal{E} \cup \mathcal{I}$ such that

$$\begin{aligned} \triangledown f(x^*) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \cdot \triangledown g_i(x^*) = 0, \\ \lambda_i \geq 0, i \in \mathcal{I}, \\ \lambda_i \cdot g_i(\mathbf{x}^*) = 0, i \in \mathcal{I}. \end{aligned}$$

First order necessary conditions

First order necessary conditions

Problem Formulation: Example

First order necessary conditions

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$$\max(x^2 + y^2)$$

First order necessary conditions

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$$\max(x^2 + y^2)$$
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First order necessary conditions

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Lagrangian function

First order necessary conditions

Problem Formulation: Example

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$$x \geq 0$$
$$y \geq 0$$
$$x + y \leq 1$$

Lagrangian function

$$L(x, y, \lambda) = -(x^2 + y^2) - \lambda_1 \cdot x - \lambda_2 \cdot y - \lambda_3 \cdot (1 - x - y).$$

First order necessary conditions

First order necessary conditions

The constraints

First order necessary conditions

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$$-2 \cdot x - \lambda_1 + \lambda_3 = 0$$

First order necessary conditions

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$$-2 \cdot x - \lambda_1 + \lambda_3 = 0$$
$$-2 \cdot y - \lambda_2 + \lambda_3 = 0$$

First order necessary conditions

The constraints

$$
\begin{aligned}
-2 \cdot x - \lambda_1 + \lambda_3 &= 0 \\
-2 \cdot y - \lambda_2 + \lambda_3 &= 0 \\
\lambda_1, \lambda_2, \lambda_3 &\geq 0
\end{aligned}
$$

First order necessary conditions

The constraints

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-2 \cdot x - \lambda_1 + \lambda_3 &= 0 \\
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\lambda_1 \cdot x &= 0
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\lambda_1 \cdot x &= 0 \\
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\lambda_3 \cdot (1 - x - y) &= 0
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Stationary points

First order necessary conditions

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$$\lambda_2 \cdot y = 0$$
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Stationary points

If $\lambda_3 = 0$, it can be shown that $x = y = 0$.

First order necessary conditions

The constraints

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-2 \cdot x - \lambda_1 + \lambda_3 &= 0 \\
-\,2 \cdot y - \lambda_2 + \lambda_3 &= 0 \\
\lambda_1, \lambda_2, \lambda_3 &\geq 0 \\
\lambda_1 \cdot x &= 0 \\
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Stationary points

If $\lambda_3 = 0$, it can be shown that $x = y = 0$.

On the other hand, if $\lambda_3 \neq 0$, then $x + y = 1$,

First order necessary conditions

The constraints

$$-2 \cdot x - \lambda_1 + \lambda_3 = 0$$
$$-2 \cdot y - \lambda_2 + \lambda_3 = 0$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$
$$\lambda_1 \cdot x = 0$$
$$\lambda_2 \cdot y = 0$$
$$\lambda_3 \cdot (1 - x - y) = 0$$

Stationary points

If $\lambda_3 = 0$, it can be shown that $x = y = 0$.

On the other hand, if $\lambda_3 \neq 0$, then $x + y = 1$, hence the problem is reduced to one dimensional case, which by standard methods lead to points $(x = 0, y = 1)$, $(x = 1, y = 0)$ and $(x = y = \frac{1}{2})$.

Second order necessary conditions

Second order necessary conditions

Theorem

Second order necessary conditions

Theorem

Assume that f and g_i, $i \in \mathcal{E} \cup \mathcal{I}$ are all twice continuously differentiable functions.

Second order necessary conditions

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Assume that f and g_i, $i \in \mathcal{E} \cup \mathcal{I}$ are all twice continuously differentiable functions.

Let $\mathbf{x}^*$ be a local minimizer of the non-linear problem,

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Theorem

Assume that f and g_i, $i \in \mathcal{E} \cup \mathcal{I}$ are all twice continuously differentiable functions.

Let $\mathbf{x}^*$ be a local minimizer of the non-linear problem, and assume that $\mathbf{x}^*$ is a regular point for the constraints of the problem.

Second order necessary conditions

Theorem

Assume that f and g_i, $i \in \mathcal{E} \cup \mathcal{I}$ are all twice continuously differentiable functions.

Let $\mathbf{x}^*$ be a local minimizer of the non-linear problem, and assume that $\mathbf{x}^*$ is a regular point for the constraints of the problem.

Then there exists $\lambda_i, i \in \mathcal{E} \cup \mathcal{I}$ satisfying the conditions of the previous theorem as well as the following condition:

Second order necessary conditions

Theorem

Assume that f and g_i, $i \in \mathcal{E} \cup \mathcal{I}$ are all twice continuously differentiable functions.

Let $\mathbf{x}^*$ be a local minimizer of the non-linear problem, and assume that $\mathbf{x}^*$ is a regular point for the constraints of the problem.

Then there exists $\lambda_i, i \in \mathcal{E} \cup \mathcal{I}$ satisfying the conditions of the previous theorem as well as the following condition:

$$\bigtriangledown^2 f(\mathbf{x}^*) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i \cdot \bigtriangledown^2 g_i(\mathbf{x}^*)$$

Is positive semidefinite on the tangent subspace of active constraints at $\mathbf{x}^*$.

Second order necessary conditions

Theorem

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Active Constraint

Second order necessary conditions

Theorem

Assume that f and g_i, $i \in \mathcal{E} \cup \mathcal{I}$ are all twice continuously differentiable functions.

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Is positive semidefinite on the tangent subspace of active constraints at $\mathbf{x}^*$.

Active Constraint

Recall that a constraint is said to be active at a point $\mathbf{x}^*$, if it satisfies the constraint with equality.

Second order necessary conditions

Second order necessary conditions

Restatement of the last condition

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Let $\mathbf{A}(\mathbf{x}^*)$ denote the Jacobian of the active constraints at $\mathbf{x}^*$,

Second order necessary conditions

Restatement of the last condition

Let $\mathbf{A}(\mathbf{x}^*)$ denote the Jacobian of the active constraints at $\mathbf{x}^*$, and let $\mathbf{N}(\mathbf{x}^*)$ be a null-space basis for $\mathbf{A}(\mathbf{x}^*)$.

Second order necessary conditions

Restatement of the last condition

Let $\mathbf{A}(\mathbf{x}^*)$ denote the Jacobian of the active constraints at $\mathbf{x}^*$, and let $\mathbf{N}(\mathbf{x}^*)$ be a null-space basis for $\mathbf{A}(\mathbf{x}^*)$.

Then, the last condition of the previous theorem, is equivalent to the following condition:

Second order necessary conditions

Restatement of the last condition

Let $\mathbf{A}(\mathbf{x}^*)$ denote the Jacobian of the active constraints at $\mathbf{x}^*$, and let $\mathbf{N}(\mathbf{x}^*)$ be a null-space basis for $\mathbf{A}(\mathbf{x}^*)$.

Then, the last condition of the previous theorem, is equivalent to the following condition:

$$\mathbf{N}^{\mathbf{T}}(\mathbf{x}^*) \cdot (\bigtriangledown^{\mathbf{2}}\mathbf{f}(\mathbf{x}^*) - \sum_{\mathbf{i} \in \mathcal{E} \cup \mathcal{I}} \lambda_{\mathbf{i}} \cdot \bigtriangledown^{\mathbf{2}}\mathbf{g}_{\mathbf{i}}(\mathbf{x}^*)) \cdot \mathbf{N}(\mathbf{x}^*)$$

Is positive semidefinite.

Second order sufficient conditions

Second order sufficient conditions

Theorem

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Theorem

Assume that f *and* g_i, $i \in \mathcal{E} \cup \mathcal{I}$ *are all twice continuously differentiable functions.*

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Second order sufficient conditions

Theorem

Assume that f *and* g_i, $i \in \mathcal{E} \cup \mathcal{I}$ *are all twice continuously differentiable functions.*

Let $\mathbf{x}^*$ *be a feasible solution of the non-linear problem, and assume that it is a regular point for the constraints of the problem.*

Second order sufficient conditions

Theorem

Assume that f and g_i, $i \in \mathcal{E} \cup \mathcal{I}$ are all twice continuously differentiable functions.

Let $\mathbf{x}^*$ be a feasible solution of the non-linear problem, and assume that it is a regular point for the constraints of the problem.

Let $\mathbf{A}(\mathbf{x}^*)$ denote the Jacobian of the active constraints at $\mathbf{x}^*$, and let $\mathbf{N}(\mathbf{x}^*)$ be a null-space basis for $\mathbf{A}(\mathbf{x}^*)$.

Second order sufficient conditions

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Assume that f and g_i, $i \in \mathcal{E} \cup \mathcal{I}$ are all twice continuously differentiable functions.

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If there exists $\lambda_i, i \in \mathcal{E} \cup \mathcal{I}$ satisfying the conditions of the first order necessary theorem as well as the following condition:

Second order sufficient conditions

Theorem

Assume that f and g_i, $i \in \mathcal{E} \cup \mathcal{I}$ are all twice continuously differentiable functions.

Let $\mathbf{x}^*$ be a feasible solution of the non-linear problem, and assume that it is a regular point for the constraints of the problem.

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If there exists $\lambda_i, i \in \mathcal{E} \cup \mathcal{I}$ satisfying the conditions of the first order necessary theorem as well as the following condition:

$$g_i(\mathbf{x}^*) = 0, i \in \mathcal{I} \text{ implies } \lambda_i > 0,$$

Second order sufficient conditions

Theorem

Assume that f and g_i, $i \in \mathcal{E} \cup \mathcal{I}$ are all twice continuously differentiable functions.

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Let $\mathbf{A}(\mathbf{x}^*)$ denote the Jacobian of the active constraints at $\mathbf{x}^*$, and let $\mathbf{N}(\mathbf{x}^*)$ be a null-space basis for $\mathbf{A}(\mathbf{x}^*)$.

If there exists $\lambda_i, i \in \mathcal{E} \cup \mathcal{I}$ satisfying the conditions of the first order necessary theorem as well as the following condition:

$$g_i(\mathbf{x}^*) = 0, i \in \mathcal{I} \text{ implies } \lambda_i > 0,$$

and

$$\mathbf{N}^{\mathbf{T}}(\mathbf{x}^*) \cdot (\bigtriangledown^2 \mathbf{f}(\mathbf{x}^*) - \sum_{\mathbf{i} \in \mathcal{E} \cup \mathcal{I}} \lambda_{\mathbf{i}} \cdot \bigtriangledown^2 \mathbf{g}_{\mathbf{i}}(\mathbf{x}^*)) \cdot \mathbf{N}(\mathbf{x}^*)$$

is positive semidefinite,

Second order sufficient conditions

Theorem

Assume that f and g_i, $i \in \mathcal{E} \cup \mathcal{I}$ are all twice continuously differentiable functions.

Let $\mathbf{x}^*$ be a feasible solution of the non-linear problem, and assume that it is a regular point for the constraints of the problem.

Let $\mathbf{A}(\mathbf{x}^*)$ denote the Jacobian of the active constraints at $\mathbf{x}^*$, and let $\mathbf{N}(\mathbf{x}^*)$ be a null-space basis for $\mathbf{A}(\mathbf{x}^*)$.

If there exists $\lambda_i, i \in \mathcal{E} \cup \mathcal{I}$ satisfying the conditions of the first order necessary theorem as well as the following condition:

$$g_i(\mathbf{x}^*) = 0, i \in \mathcal{I} \textit{ implies } \lambda_i > 0,$$

and

$$\mathbf{N^T}(\mathbf{x}^*) \cdot (\bigtriangledown^2 \mathbf{f}(\mathbf{x}^*) - \sum_{\mathbf{i} \in \mathcal{E} \cup \mathcal{I}} \lambda_\mathbf{i} \cdot \bigtriangledown^2 \mathbf{g_i}(\mathbf{x}^*)) \cdot \mathbf{N}(\mathbf{x}^*)$$

is positive semidefinite, then $\mathbf{x}^*$ is local minimizer of the non-linear program.

Constrained non-linear programs

Constrained non-linear programs

An approach

Constrained non-linear programs

An approach

Below, we introduce an approach for solving non-linear programs.

Constrained non-linear programs

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Below, we introduce an approach for solving non-linear programs.

It relies on the method of steepest decent method.

Constrained non-linear programs

An approach

Below, we introduce an approach for solving non-linear programs.

It relies on the method of steepest decent method.

The idea is to reduce the number of variables using the constraints and to solve this reduced and unconstrained problem using the steepest decent method.

Linear equality constraints

Linear equality constraints

Problem Formulation

Linear equality constraints

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$$\min f(\mathbf{x}) := x_1^3 + x_2 + x_3^3 + x_4$$

Linear equality constraints

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$$\min f(\mathbf{x}) := x_1^3 + x_2 + x_3^3 + x_4$$
$$g_1(\mathbf{x}) := x_1 + x_2 + 4 \cdot x_3 + 4 \cdot x_4 - 4 = 0,$$

Linear equality constraints

Problem Formulation

$$\min f(\mathbf{x}) := x_1^3 + x_2 + x_3^3 + x_4$$
$$g_1(\mathbf{x}) := x_1 + x_2 + 4 \cdot x_3 + 4 \cdot x_4 - 4 = 0,$$
$$g_2(\mathbf{x}) := -x_1 + x_2 + 2 \cdot x_3 - 2 \cdot x_4 + 2 = 0,$$

Linear equality constraints

Linear equality constraints

Solving the linear equations

Linear equality constraints

Solving the linear equations

$$x_2 = 3 \cdot x_1 + 8 \cdot x_4 - 8,$$

Linear equality constraints

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$$x_2 = 3 \cdot x_1 + 8 \cdot x_4 - 8,$$
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Linear equality constraints

Linear equality constraints

Solving an unconstrained non-linear program

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The above equations lead to the following unconstrained non-linear program:

Linear equality constraints

Solving an unconstrained non-linear program

The above equations lead to the following unconstrained non-linear program:

$$\min f(x_1, x_4) := x_1^3 + (3 \cdot x_1 + 8 \cdot x_4 - 8) + (-x_1 - 3 \cdot x_4 + 3)^3 + x_4.$$

Non-linear equality constraints

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Solving an unconstrained non-linear program

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In this example, the constraints are non-linear.

Non-linear equality constraints

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Taylor series

Non-linear equality constraints

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Approximate the constraint functions by their Taylor series:

Non-linear equality constraints

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Approximate the constraint functions by their Taylor series:
$\mathbf{g}(\mathbf{x}) \approx \mathbf{g}(\bar{\mathbf{x}}) + \bigtriangledown \mathbf{g}(\bar{\mathbf{x}}) \cdot (\mathbf{x} - \bar{\mathbf{x}})^T$,

Non-linear equality constraints

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$\mathbf{g}(\mathbf{x}) \approx \mathbf{g}(\bar{\mathbf{x}}) + \bigtriangledown \mathbf{g}(\bar{\mathbf{x}}) \cdot (\mathbf{x} - \bar{\mathbf{x}})^T$, where $\bar{\mathbf{x}}$ is the current point.

Non-linear equality constraints

Non-linear equality constraints

Our constraints become

Non-linear equality constraints

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$$g_1(\mathbf{x}) \approx 2 \cdot \bar{x}_1 \cdot x_1 + x_2 + 4 \cdot x_3 + 4 \cdot x_4 - (\bar{x}_1^2 + 4) = 0,$$

Non-linear equality constraints

Our constraints become

$$g_1(\mathbf{x}) \approx 2 \cdot \bar{x}_1 \cdot x_1 + x_2 + 4 \cdot x_3 + 4 \cdot x_4 - (\bar{x}_1^2 + 4) = 0,$$
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Non-linear equality constraints

Non-linear equality constraints

The general idea

Non-linear equality constraints

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The idea of Generalized Reduced Gradient Method (GRG) is to solve a sequence of sub-problems, each of which uses a linear approximation of the constraints.

Non-linear equality constraints

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In each iteration of the algorithm, the constraint linearization is recalculated at the point found from the previous iteration.

Non-linear equality constraints

The general idea

The idea of Generalized Reduced Gradient Method (GRG) is to solve a sequence of sub-problems, each of which uses a linear approximation of the constraints.

In each iteration of the algorithm, the constraint linearization is recalculated at the point found from the previous iteration.

Though the constraints are only approximated, the subproblems yield points that are progressively closer to the optimal point.

Non-linear equality constraints

Non-linear equality constraints

Our example: Starting point

Non-linear equality constraints

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Let us start with $\mathbf{x}^0 = (0, -8, 3, 0)$.

Non-linear equality constraints

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Let us start with $\mathbf{x}^0 = (0, -8, 3, 0)$.

This point satisfies our constraints.

Non-linear equality constraints

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It is quite possible to start with an infeasible point.

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Using the approximation formulas derived earlier, we get:

Non-linear equality constraints

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Non-linear equality constraints

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Non-linear equality constraints

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Non-linear equality constraints

Non-linear equality constraints

The resulting unconstrained program

Non-linear equality constraints

The resulting unconstrained program

$$\min f(x_1, x_4) := x_1^2 + (2 \cdot x_1 + 4 \cdot x_4 - 8) + (-\frac{1}{2} \cdot x_1 - 2 \cdot x_4 + 3)^2 + x_4.$$

Non-linear equality constraints

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The next point

Non-linear equality constraints

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The next point

Solving this unconstrained minimization problem, we get $x_1 = -0.375$ and $x_4 = 0.96875$.

Non-linear equality constraints

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Solving this unconstrained minimization problem, we get $x_1 = -0.375$ and $x_4 = 0.96875$.

Substituting in equations for x_2 and x_3 gives $x_2 = -4.875$ and $x_3 = 1.25$.

Non-linear equality constraints

The resulting unconstrained program

$$\min f(x_1, x_4) := x_1^2 + (2 \cdot x_1 + 4 \cdot x_4 - 8) + (-\frac{1}{2} \cdot x_1 - 2 \cdot x_4 + 3)^2 + x_4.$$

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Solving this unconstrained minimization problem, we get $x_1 = -0.375$ and $x_4 = 0.96875$.

Substituting in equations for x_2 and x_3 gives $x_2 = -4.875$ and $x_3 = 1.25$.

Thus, the iteration of GRG method is $\mathbf{x}^1 = (-0.375, -4.875, 1.25, 0.96875)$.

Non-linear equality constraints

Non-linear equality constraints

Continuing the process

Non-linear equality constraints

Continuing the process

To continue the solution process, we would re-linearize the constraint functions at the new point,

Non-linear equality constraints

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To continue the solution process, we would re-linearize the constraint functions at the new point,

Use the resulting system of linear equations to express two of the variables in terms of the others,

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Solve the reduced problem for $\mathbf{x}^2$, and so forth.

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Stopping criterion

Non-linear equality constraints

Continuing the process

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Stopping criterion

The stopping criterion is $\|\mathbf{x}^{k+1} - \mathbf{x}^k\| < T$, where T is a small constant.

Non-linear equality constraints

Non-linear equality constraints

Our example

Non-linear equality constraints

Our example

For example if we take, $T = 0.0025$ in the above example, we get $\mathbf{x}^k = (-0.498, -4, 823, 1.534, 0.610)$ and $f(\mathbf{x}^k) = -1.612$.

Non-linear equality constraints

Our example

For example if we take, $T = 0.0025$ in the above example, we get $\mathbf{x}^k = (-0.498, -4, 823, 1.534, 0.610)$ and $f(\mathbf{x}^k) = -1.612$.

The optimum solution is $\mathbf{x}^* = (-0.500, -4.825, 1.534, 0.610)$ and has an objective value of -1.612.

Non-linear equality constraints

Non-linear equality constraints

Remark

Non-linear equality constraints

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During the iteration, the values of $f(\mathbf{x}^k)$ can sometimes be smaller than the minimum value.

Non-linear equality constraints

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Non-linear equality constraints

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The reason is that the points $\mathbf{x}^k$ computed by GRG are usually not feasible.

Non-linear equality constraints

Remark

During the iteration, the values of $f(\mathbf{x}^k)$ can sometimes be smaller than the minimum value.

How this is possible?

The reason is that the points $\mathbf{x}^k$ computed by GRG are usually not feasible.

They are only feasible for linear approximations of these constraints.

Non-linear equality constraints

Non-linear equality constraints

Starting from an infeasible solution

Non-linear equality constraints

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Assume that we have chosen the point $\mathbf{x}^0$ so that it is infeasible.

Non-linear equality constraints

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We consider a phase 1 problem, which is the construction of a feasible solution.

Non-linear equality constraints

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The objective function for the phase 1 problem is the sum of the absolute values of the violated constraints.

Non-linear equality constraints

Starting from an infeasible solution

Assume that we have chosen the point $\mathbf{x}^0$ so that it is infeasible.

We consider a phase 1 problem, which is the construction of a feasible solution.

The objective function for the phase 1 problem is the sum of the absolute values of the violated constraints.

The constraints for the phase 1 problem are the non-violated ones.

Non-linear equality constraints

Non-linear equality constraints

Starting from an infeasible solution: Our example

Non-linear equality constraints

Starting from an infeasible solution: Our example

If we had started from the point $\mathbf{x}^0 = (1, 1, 0, 1)$, which happens to be infeasible, then the phase 1 problem would be the following:

Non-linear equality constraints

Starting from an infeasible solution: Our example

If we had started from the point $\mathbf{x}^0 = (1, 1, 0, 1)$, which happens to be infeasible, then the phase 1 problem would be the following:

$$\min |x_1^2 + x_2 + 4 \cdot x_3 + 4 \cdot x_4 - 4|$$

Non-linear equality constraints

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$$-x_1 + x_2 + 2 \cdot x_3 - 2 \cdot x_4^2 + 2 = 0,$$

Non-linear equality constraints

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If we had started from the point $\mathbf{x}^0 = (1, 1, 0, 1)$, which happens to be infeasible, then the phase 1 problem would be the following:

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This is because $\mathbf{x}^0$ violates the first constraint and satisfies the second one.

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$$-x_1 + x_2 + 2 \cdot x_3 - 2 \cdot x_4^2 + 2 = 0,$$

This is because $\mathbf{x}^0$ violates the first constraint and satisfies the second one.

Observe that the value of the objective function is 0, if and only if the corresponding point is a feasible solution.

Linear inequality constraints

Linear inequality constraints

The general strategy

Linear inequality constraints

The general strategy

We will discuss how GRG solves problems when there are inequality as well as equality constraints.

Linear inequality constraints

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At each iteration, only the tight inequality constraints enter into the system of linear linear equations used for eliminating variables (active inequality constraints).

Linear inequality constraints

The general strategy

We will discuss how GRG solves problems when there are inequality as well as equality constraints.

At each iteration, only the tight inequality constraints enter into the system of linear linear equations used for eliminating variables (active inequality constraints).

The process is complicated by the fact that active inequality constraints at the current point may need to be released in order to move to a better solution.

Linear inequality constraints

Linear inequality constraints

An example

Linear inequality constraints

An example

$$\min f(x_1, x_2) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{5}{2})^2$$

Linear inequality constraints

An example

$$\min f(x_1, x_2) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{5}{2})^2$$
$$x_1 - x_2 \geq 0,$$

Linear inequality constraints

An example

$$\min f(x_1, x_2) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{5}{2})^2$$
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Linear inequality constraints

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$$\min f(x_1, x_2) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{5}{2})^2$$
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$$x_1 \geq 0,$$
$$x_2 \geq 0,$$
$$x_2 \leq 2,$$

Linear inequality constraints

An example

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The process

Linear inequality constraints

An example

$$\min f(x_1, x_2) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{5}{2})^2$$
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$$x_1 \geq 0,$$
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The process

Assume that the initial feasible solution is $\mathbf{x}^0 = (1, 0)$.

Linear inequality constraints

An example

$$\min f(x_1, x_2) = (x_1 - \frac{1}{2})^2 + (x_2 - \frac{5}{2})^2$$
$$x_1 - x_2 \geq 0,$$
$$x_1 \geq 0,$$
$$x_2 \geq 0,$$
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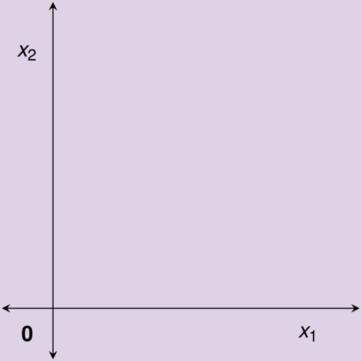
The process

Assume that the initial feasible solution is $\mathbf{x}^0 = (1, 0)$.

It can be checked directly that the constraints $x_1 - x_2 \geq 0$, $x_1 \geq 0$, and $x_2 \leq 2$ are inactive, whereas the constraint $x_2 \geq 0$ is active.

The feasible region



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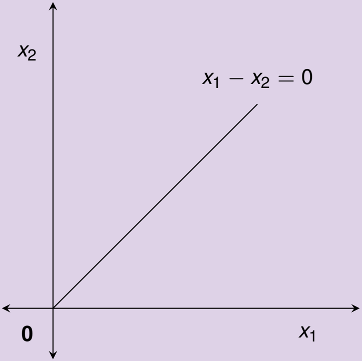

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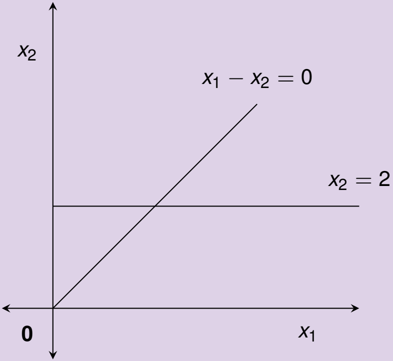

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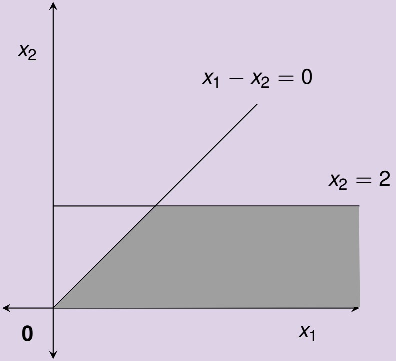

Linear inequality constraints

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First let us evaluate the gradient of the objective function at $\mathbf{x}^0 = (1, 0)$.

Linear inequality constraints

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First let us evaluate the gradient of the objective function at $\mathbf{x}^0 = (1, 0)$.

$$\bigtriangledown f(\mathbf{x}^0) = (2 \cdot x_1^0 - 1, 2 \cdot x_2^0 - 5) = (1, -5).$$

Linear inequality constraints

An example

First let us evaluate the gradient of the objective function at $\mathbf{x}^0 = (1, 0)$.

$$\bigtriangledown f(\mathbf{x}^0) = (2 \cdot x_1^0 - 1, 2 \cdot x_2^0 - 5) = (1, -5).$$

This means that we will get the largest decrease in f if we move in the direction $\mathbf{d^0} = -\bigtriangledown \mathbf{f}(\mathbf{x^0}) = (-\mathbf{1}, \mathbf{5})$, that is, if we decrease x_1 and increase x_2.

Linear inequality constraints

An example

First let us evaluate the gradient of the objective function at $\mathbf{x}^0 = (1, 0)$.

$$\bigtriangledown f(\mathbf{x}^0) = (2 \cdot x_1^0 - 1, 2 \cdot x_2^0 - 5) = (1, -5).$$

This means that we will get the largest decrease in f if we move in the direction $\mathbf{d^0} = -\bigtriangledown \mathbf{f}(\mathbf{x^0}) = (-\mathbf{1}, \mathbf{5})$, that is, if we decrease x_1 and increase x_2.

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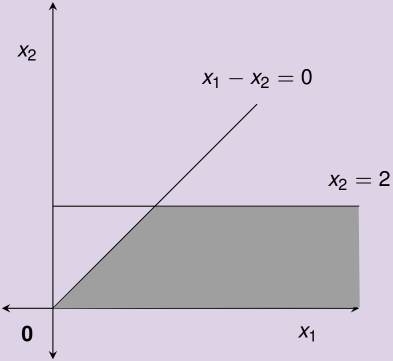
It can be shown that $\alpha^0 = 0.8333$, so $\mathbf{x}^1 = (0.8333, 0.8333)$.

The feasible region and the progress of the algorithm



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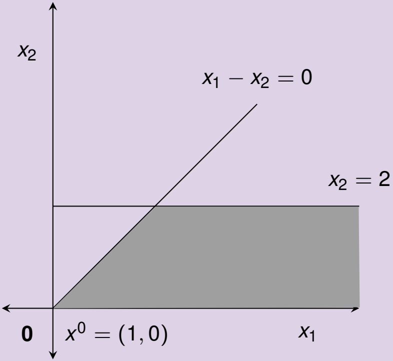

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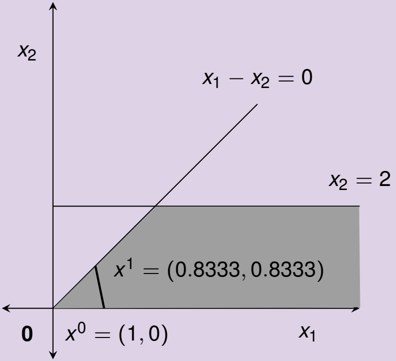

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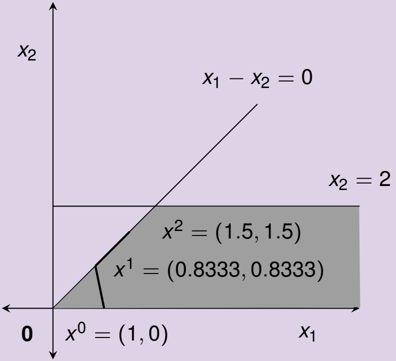

Non-linear programming

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Problem Formulation

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Consider a general non-linear optimization problem:

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In order to solve this problem, we can use methods available for quadratic programs.

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This is the idea behind sequential quadratic programming.

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At the current feasible point $\mathbf{x}^k$, the problem is approximated by a quadratic program:

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Where $\mathbf{B_k} = \bigtriangledown^2_{\mathbf{xx}} \mathbf{L}(\mathbf{x^k}, \lambda^\mathbf{k})$ is the Hessian of the Langrangian function with respect to the variables x and λ^k is the current estimate of the Lagrangian multipliers.

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Then a line search is performed to determine the next iterate.

Non-smooth Optimization

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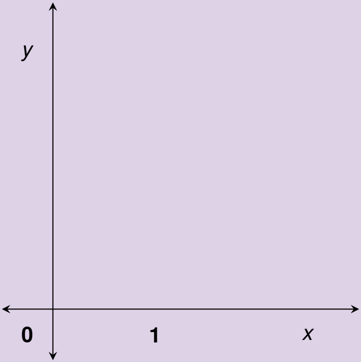
For example consider the function

$$f(x) = |x - 1|.$$

The subgradients of the function $y = |x - 1|$



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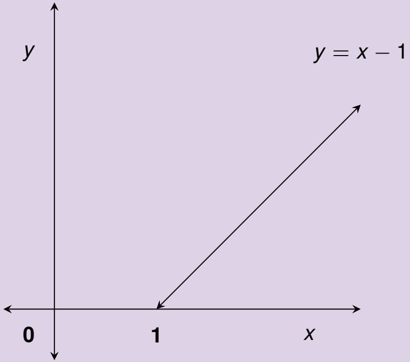

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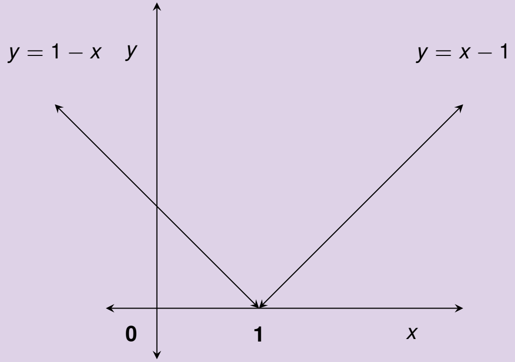

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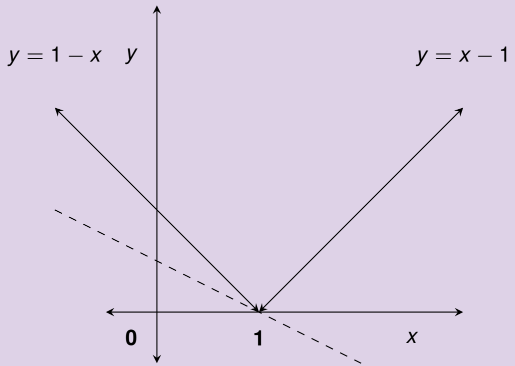

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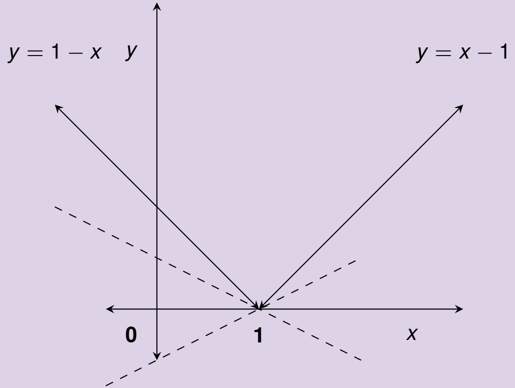

A property of convex functions

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Example

In the case of $f(x) = |x - 1|$, 0 is a subgradient at point $x^* = 1$, therefore this is the point where the minimum of f is achieved.

The method of steepest decent for convex functions

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First we compute any subgradient direction at the current point, and use its opposite direction to make the next step.

The method of steepest decent for convex functions

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The method of steepest decent can be extended to non-differentiable functions.

First we compute any subgradient direction at the current point, and use its opposite direction to make the next step.

Though subgradient directions are not always directions of ascent, one can still guarantee convergence to the optimum point by choosing the step size appropriately.

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Formal description of the method

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For example, for the choice of $\alpha_i \to 0$ such that $\sum_i \alpha_i = +\infty$, will do.

Literature

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The list of references

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