Optimization Methods in Finance - Homework I

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1 Instructions

1. The homework is due on March 10.
2. Each question is worth 4 points.
3. Attempt as many problems as you can. You will be given partial credit, as per the policy discussed in class.

2 Problems

1. **Convexity:**

   (a) Let S denote a set and let $\mathbf{x}$ denote an extreme point of S. Argue that $\mathbf{x}$ is an extreme point of S, if and only if $S - \{\mathbf{x}\}$ is convex.

   (b) Consider the linear program:

$$\begin{array}{rcl} & \max \mathbf{c} \cdot \mathbf{x} & \\ \mathbf{A} \cdot \mathbf{x} & \leq & \mathbf{b} \\ \mathbf{x} & \geq & \mathbf{0} \end{array}$$

   Let $\mathbf{x_1}$ and $\mathbf{x_2}$ represent two optimal solutions for the above linear program. Argue that the parametric point $(\alpha \cdot \mathbf{x_1} + (1-\alpha) \cdot \mathbf{x_2})$, $\alpha \in [0, 1]$ is also an optimal solution for the linear program.

   (c) Given two convex sets S_1 and S_2, what can you say about the sets $S_1 \cup S_2$ and $S_1 \cap S_2$ as regards convexity.

2. **Linear Programming:**

   Consider the following linear program:

$$\begin{array}{rcl} \max z = x_1 + 2 \cdot x_2 - 9 \cdot x_3 + 8 \cdot x_4 - 36 \cdot x_5 & & \\ 2 \cdot x_2 - x_3 + x_4 - 3 \cdot x_5 & \leq & 40 \\ x_1 - x_2 + 2 \cdot x_4 - 2 \cdot x_5 & \leq & 10 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

   (a) Solve the problem using the Simplex algorithm, showing all the steps.

   (b) Write down the dual of the above problem.

   (c) Solve the dual graphically and then solve the primal using complementary slackness.

3. **Non-linear Programming (Theory):**

   (a) Solve the following NLP analitically:

$$\min(x_2 + x_3 - 5)^2 - \frac{(x_3 + x_4 + 2)}{7} + 4$$
$$-x_1 + x_2 + x_3 - 5 = 0$$
$$-3 \cdot x_1 - x_2 + x_3 - 3 = 0$$
$$-7 \cdot x_1 + x_3 + x_4 + 2 = 0$$
$$x_1 \leq 1, x_2 \leq 1, x_3 \leq 6, x_4 \leq 1$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.$$

   (b) Check the point $(1, 2)$ for first order necessary conditions (KKT conditions):

$$\max 2 \cdot x_1^3 + 3 \cdot x_2^4$$
$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 3$$
$$x_2 - x_1 \leq 1$$
$$x_1 - x_2 \geq -1$$
$$x_1 \geq 0, x_2 \geq 0.$$

4. **Non-linear Programming (Applications):**

   The partial derivative $\frac{\partial f(x)}{\partial x_i}$ of the function $f(x)$ with respect to the ith coordinate of the x vector can be estimated as

$$\frac{\partial f(x)}{\partial x_i} \approx \frac{f(x + h \cdot e_i) - f(x)}{h},$$

   where e_i denotes the ith unit vector. Assuming that f is continuously differentiable, provide an upper bound on the estimation error from this finite-difference approximation using a Taylor series expansion for the function f around x. Next compute a similar bound for the alternative finite-difference formula given by

$$\frac{\partial f(x)}{\partial x_i} \approx \frac{f(x + h \cdot e_i) - f(x - h \cdot e_i)}{2 \cdot h}.$$

   Comment on potential advantages and disadvantages of these two approaches.

5. **Quadratic Programming:**

   (a) Consider the quadratic function $f(x) = c^T \cdot x + \frac{1}{2} \cdot x^T \cdot Q \cdot x$, where the matrix Q is $n \times n$ and symmetric.

      i. Prove that if $x^T \cdot Q \cdot x < 0$ for some x, then f is unbounded below.

      ii. Prove that if Q is positive semidefinite (but not positive definite), then either f is unbounded below or it has an infinite number of solutions.

      iii. True or False: f has a unique minimizer if and only if Q is positive definite.

   (b) Consider the following quadratic program:

$$\begin{aligned} \min \quad & x_1 \cdot x_2 + x_1^2 + \tfrac{3}{2}{x_2}^2 + 2 \cdot x_3^2 \\ & +2 \cdot x_1 + x_2 + 3 \cdot x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1 - x_2 = 0 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

      i. Is the optimization function convex?

      ii. Is the point $(\frac{1}{2}, \frac{1}{2}, 0)$ optimal? Provide a rigorous argument for your answer.